Deterministic and Ensemble-Based Spatially-Coupled Product Codes

Christian Häger[1,2] Henry D. Pfister[2]
Alexandre Graell i Amat[1] Fredrik Brännström[1]

[1] Department of Electrical Engineering, Chalmers University of Technology, Gothenburg

[2] Department of Electrical and Computer Engineering, Duke University, Durham

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Motivation and Outline

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In This Talk . . .

- Deterministic code construction that recovers product codes, staircase codes, and block-wise braided codes as special cases
- Rigorous density evolution analysis possible over the binary erasure channel
- Application: Spatially-coupled product codes

Introduction: Product Codes and Staircase Codes

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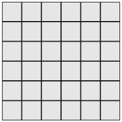
rectangular array [Elias, 1954]



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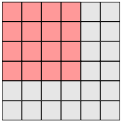

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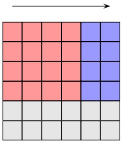

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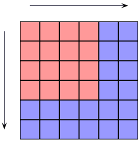

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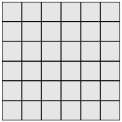

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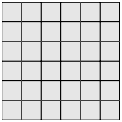

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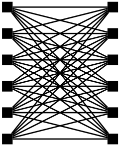
Tanner graph



constraint node degree = component code length

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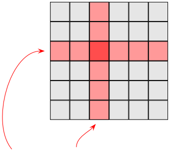

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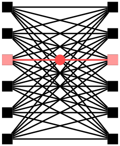

edge = degree-2 variable node

constraint node degree = component code length

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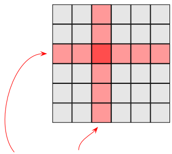

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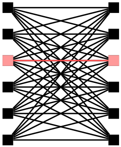

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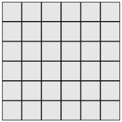

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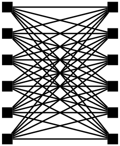

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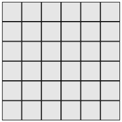

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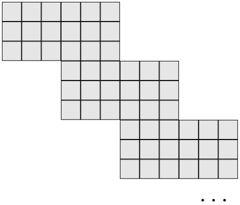

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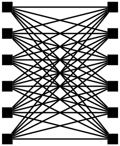

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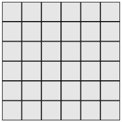

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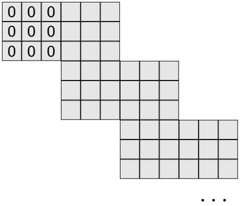

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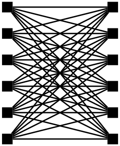

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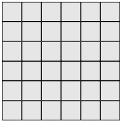

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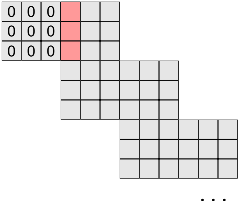

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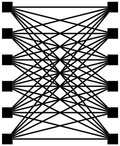

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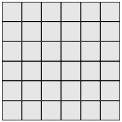

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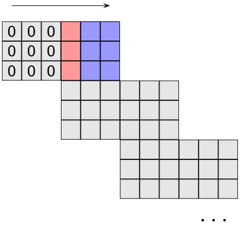

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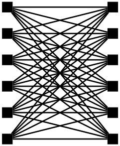

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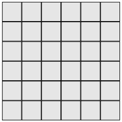

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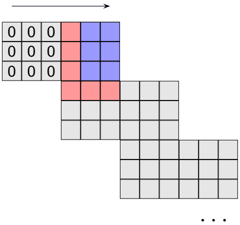

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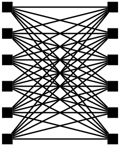

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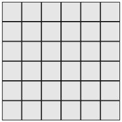

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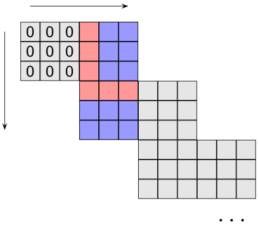

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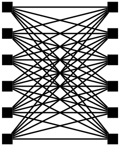

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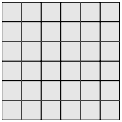

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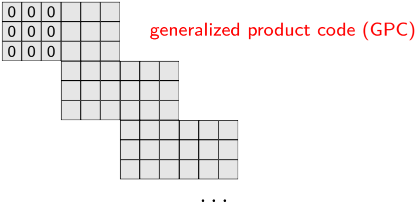

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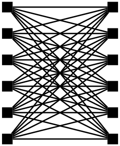

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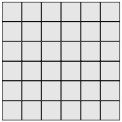

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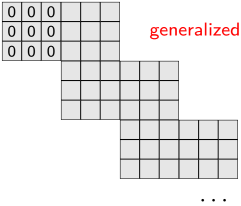
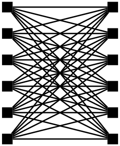
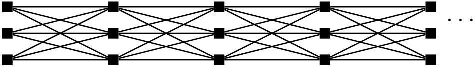


generalized product code (GPC)

Tanner graph





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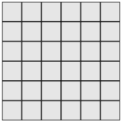

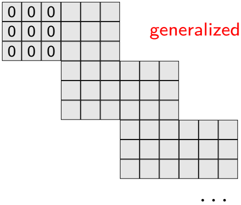

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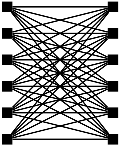
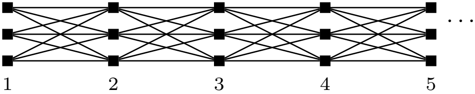


spatially-coupled code



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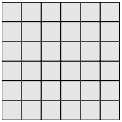

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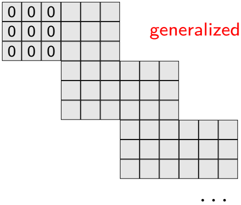

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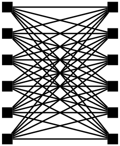

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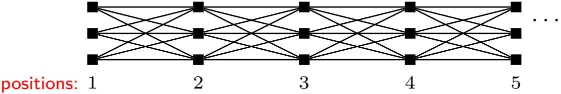
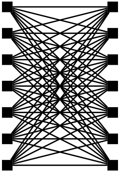
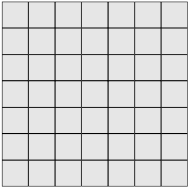


- Deterministic codes with fixed and structured Tanner graph

Iterative Bounded-Distance Decoding





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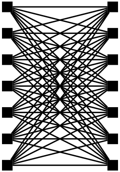

0	1	0	1	0	1	0
0	1	0	1	1	0	1
0	1	0	1	0	1	0
1	1	1	0	1	1	0
0	0	1	0	0	0	1
1	0	0	0	1	1	1
0	1	0	0	0	1	1

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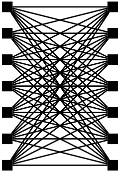
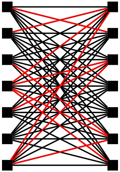


0	?	0	?	0	1	?
?	1	0	1	1	0	1
0	1	0	?	0	?	?
1	1	1	0	1	1	0
0	0	1	0	0	0	1
1	0	?	?	1	1	?
0	1	0	?	0	1	1

- Codeword transmission over binary erasure channel with erasure probability p

Iterative Bounded-Distance Decoding



0	?	0	?	0	1	?
?	1	0	1	1	0	1
0	1	0	?	0	?	?
1	1	1	0	1	1	0
0	0	1	0	0	0	1
1	0	?	?	1	1	?
0	1	0	?	0	1	1

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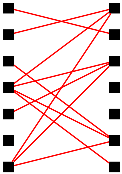

residual graph

0	?	0	?	0	1	?
?	1	0	1	1	0	1
0	1	0	?	0	?	?
1	1	1	0	1	1	0
0	0	1	0	0	0	1
1	0	?	?	1	1	?
0	1	0	?	0	1	1

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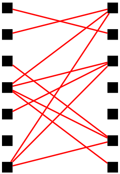

residual graph

0	?	0	?	0	1	?
?	1	0	1	1	0	1
0	1	0	?	0	?	?
1	1	1	0	1	1	0
0	0	1	0	0	0	1
1	0	?	?	1	1	?
0	1	0	?	0	1	1

- Codeword transmission over binary erasure channel with erasure probability p
- Each constraint node corresponds to t-erasure correcting component code

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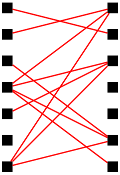

residual graph

0	?	0	?	0	1	?
?	1	0	1	1	0	1
0	1	0	?	0	?	?
1	1	1	0	1	1	0
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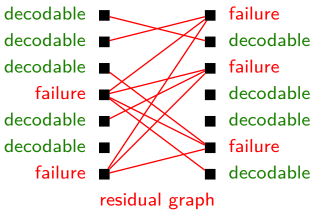
- Codeword transmission over binary erasure channel with erasure probability p
- Each constraint node corresponds to t-erasure correcting component code
- ℓ iterations of bounded-distance decoding $=$ peeling of vertices with degree $\leq t$ (in parallel)

Iterative Bounded-Distance Decoding

1st iteration $(t = 2)$



0	?	0	?	0	1	?
?	1	0	1	1	0	1
0	1	0	?	0	?	?
1	1	1	0	1	1	0
0	0	1	0	0	0	1
1	0	?	?	1	1	?
0	1	0	?	0	1	1

- Codeword transmission over binary erasure channel with erasure probability p
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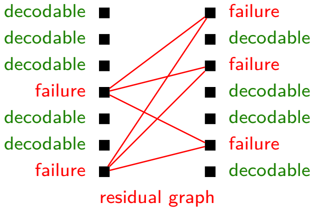

0	1	0	?	0	1	?
0	1	0	1	1	0	1
0	1	0	?	0	1	?
1	1	1	0	1	1	0
0	0	1	0	0	0	1
1	0	0	?	1	1	?
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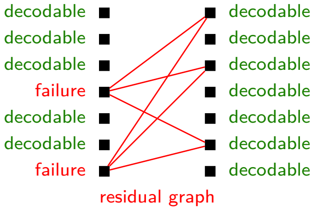
Iterative Bounded-Distance Decoding

2nd iteration $(t = 2)$



0	1	0	?	0	1	?
0	1	0	1	1	0	1
0	1	0	?	0	1	?
1	1	1	0	1	1	0
0	0	1	0	0	0	1
1	0	0	?	1	1	?
0	1	0	0	0	1	1

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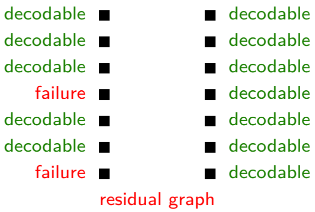

0	1	0	1	0	1	0
0	1	0	1	1	0	1
0	1	0	1	0	1	0
1	1	1	0	1	1	0
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Performance Prediction

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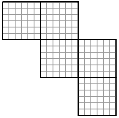

- Example: staircase code with a fixed component code

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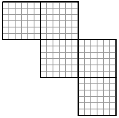

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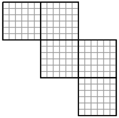

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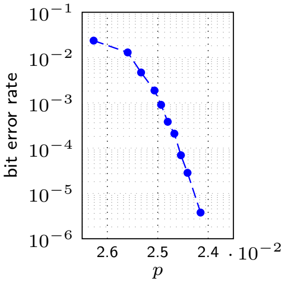

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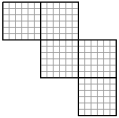

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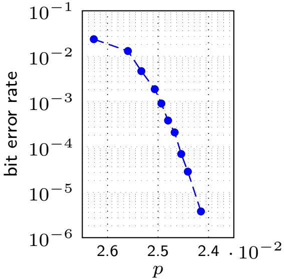

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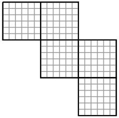

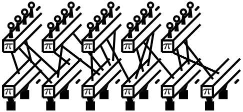

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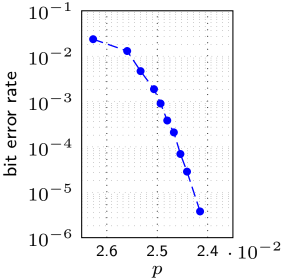

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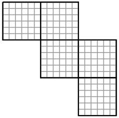

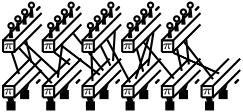

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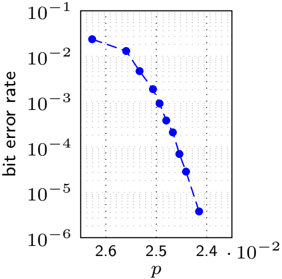

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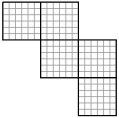

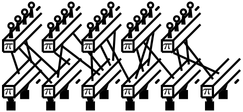

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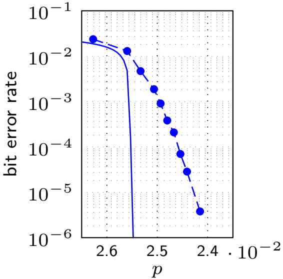

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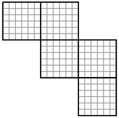

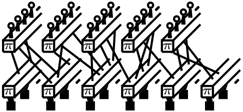

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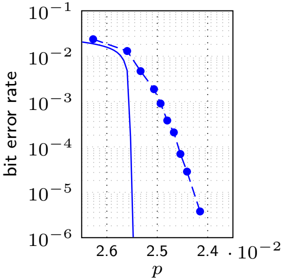
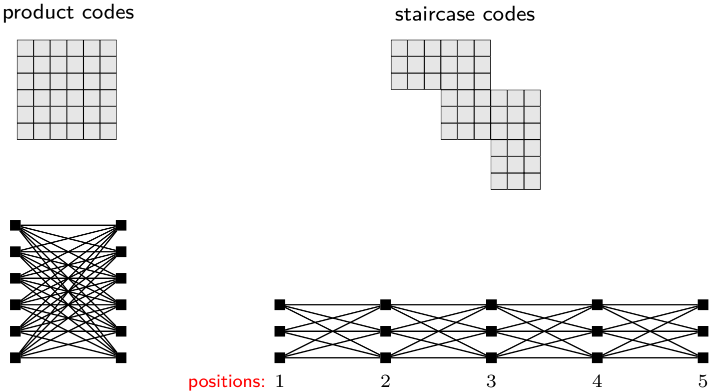


Fundamental question

Is it possible to directly analyze deterministic GPCs without the detour to code ensembles?

Parametrized Construction of Generalized Product Codes



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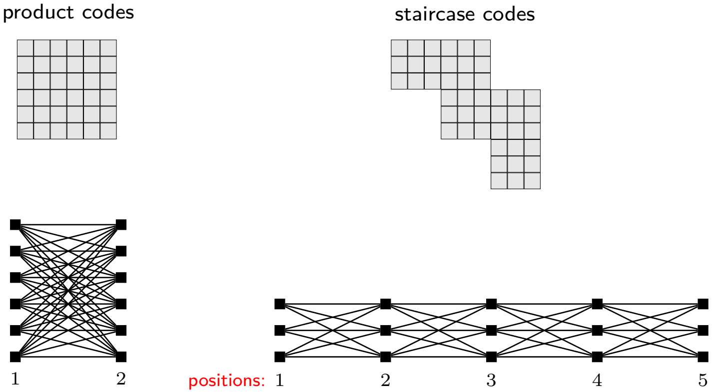

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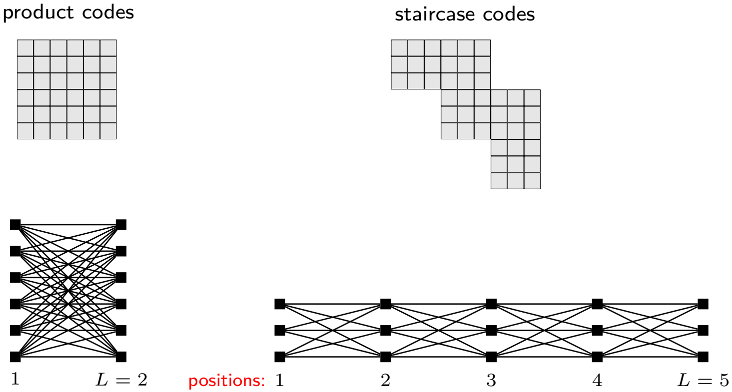
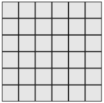
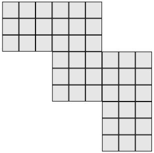


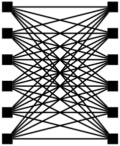
Parametrized Construction of Generalized Product Codes

product codes

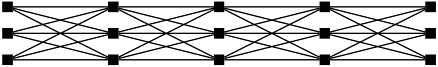


staircase codes





1 $L = 2$

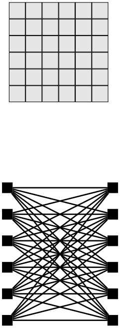


positions: 1 2 3 4 $L = 5$

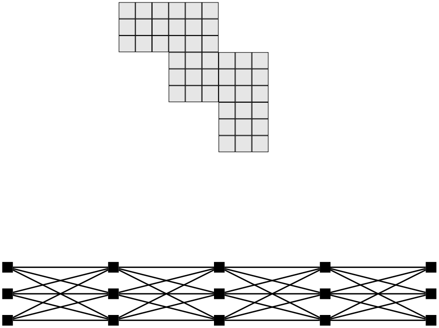
$\boldsymbol{\eta}$: symmetric $L \times L$ matrix that defines graph connectivity

Parametrized Construction of Generalized Product Codes

product codes



staircase codes



$$\boldsymbol{\eta} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

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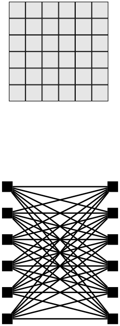
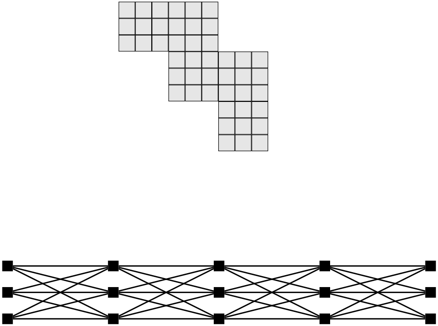
Parametrized Construction of Generalized Product Codes

product codes



$1 \qquad L = 2$

staircase codes



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Parametrized Construction of Generalized Product Codes

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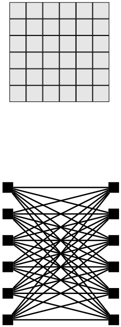

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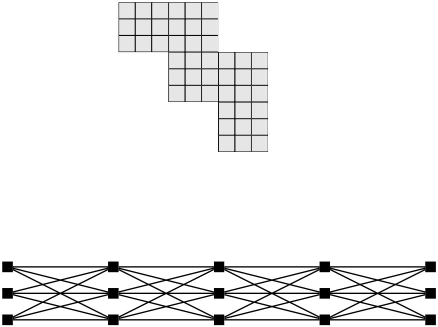

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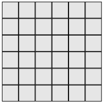

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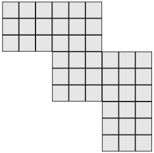
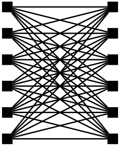
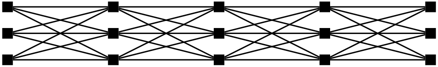




n: "problem size", proportional to the number of constraint nodes



Parametrized Construction of Generalized Product Codes

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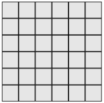

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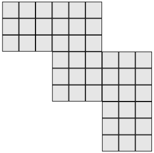

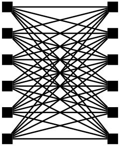

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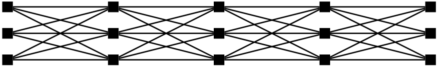
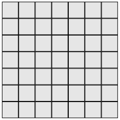


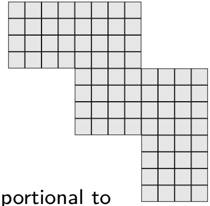
increasing n

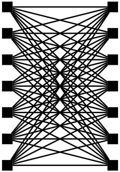
Parametrized Construction of Generalized Product Codes

product codes



staircase codes





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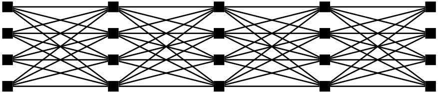

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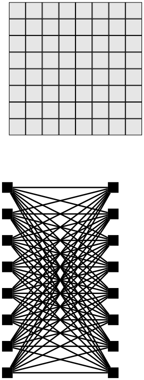

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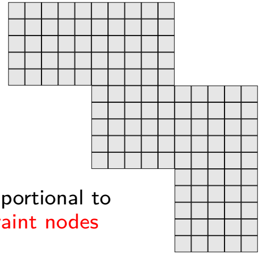

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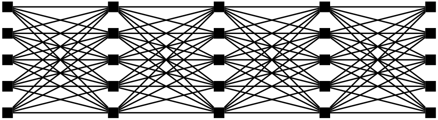

increasing n

Density Evolution

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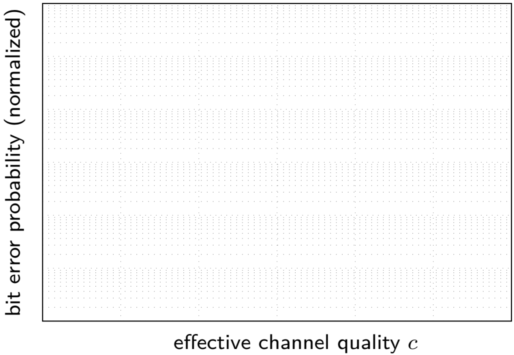

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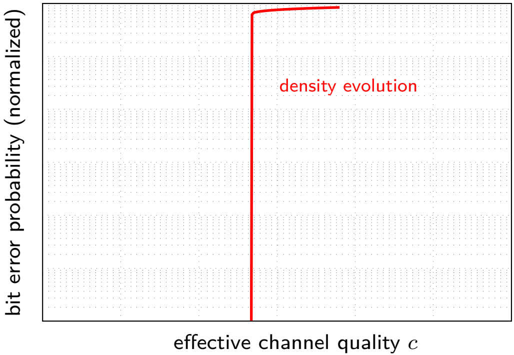

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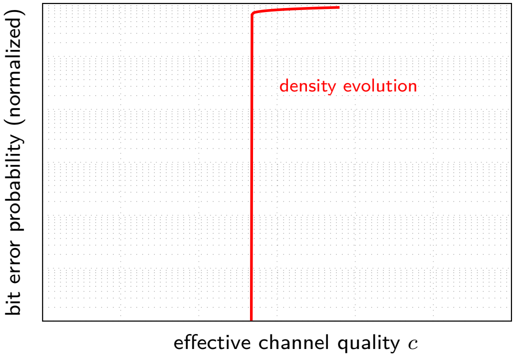

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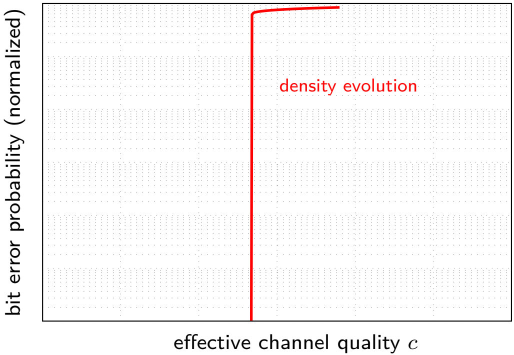

initial condition
$\boldsymbol{x}^{(0)} = (1, \ldots, 1)$

$$\boldsymbol{x}^{(\ell)} = \boldsymbol{\Psi}_{\geq t}(c\boldsymbol{B}\boldsymbol{x}^{(\ell-1)})$$

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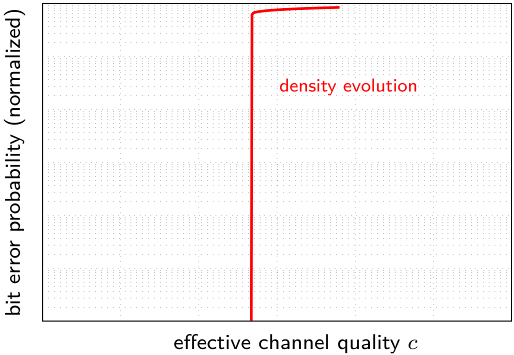

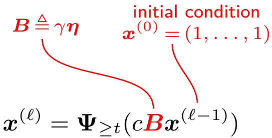

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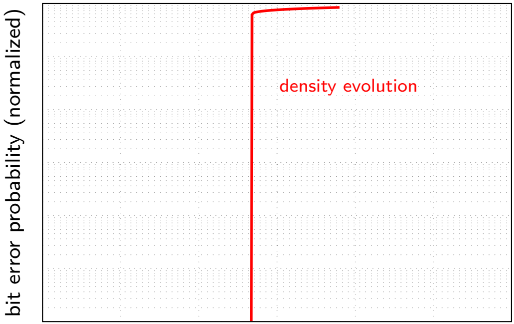
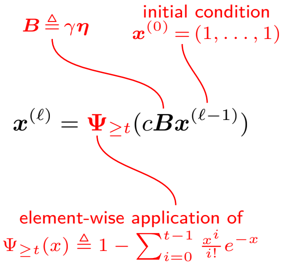
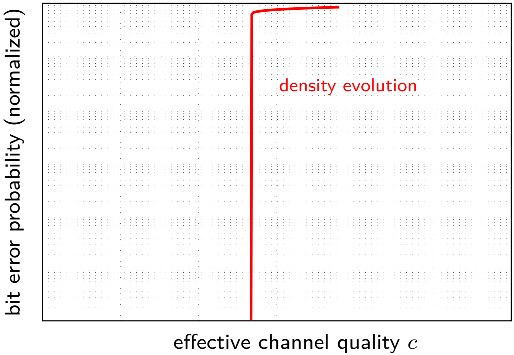
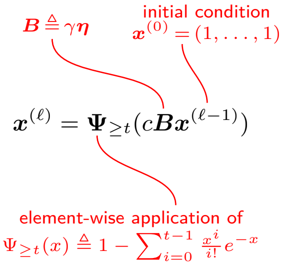




Density Evolution

- What happens asymptotically for $n \to \infty$?
- Let $p = c/n$ for $c > 0$, where c is the effective channel quality





Density Evolution

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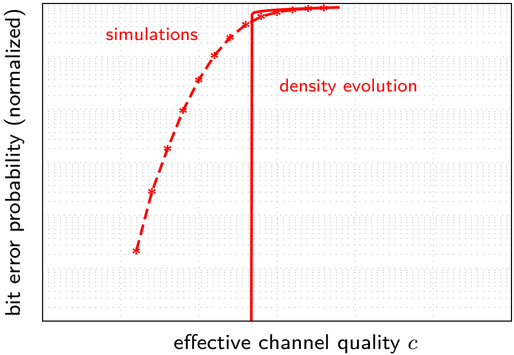

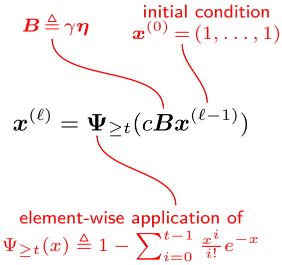

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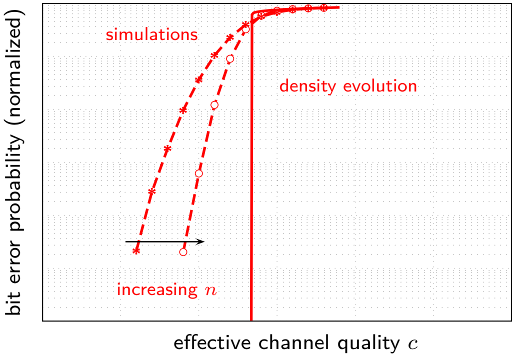

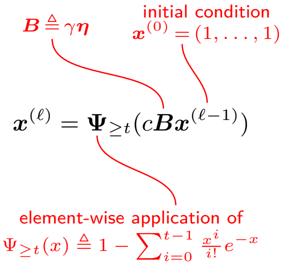

Density Evolution

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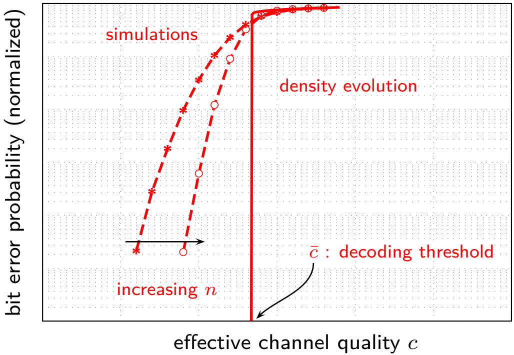

$\boldsymbol{B} \triangleq \gamma \boldsymbol{\eta}$

initial condition $\boldsymbol{x}^{(0)} = (1, \dots, 1)$

$$\boldsymbol{x}^{(\ell)} = \boldsymbol{\Psi}_{\geq t}(c\boldsymbol{B}\boldsymbol{x}^{(\ell-1)})$$

element-wise application of $\Psi_{\geq t}(x) \triangleq 1 - \sum_{i=0}^{t-1} \frac{x^i}{i!} e^{-x}$

Density Evolution

Summary

Density Evolution

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- Parametrized code construction based on η recovers many existing code classes as special cases (product codes, staircase codes, and others)

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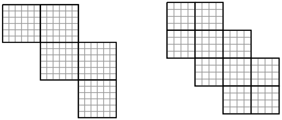
Summary

- Parametrized code construction based on η recovers many existing code classes as special cases (product codes, staircase codes, and others)
- Rigorous density evolution analysis possible over the binary erasure channel
- enables (asymptotic) performance prediction, code comparison via thresholds, efficient parameter optimization, ...

Comparison of Deterministic Codes and Ensembles

Comparison of Deterministic Codes and Ensembles

Deterministic



$$\boldsymbol{x}^{(\ell)} = \boldsymbol{\Psi}_{\geq t}(c\boldsymbol{B}\boldsymbol{x}^{(\ell-1)})$$

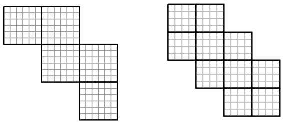
$(\boldsymbol{B} = \gamma\boldsymbol{\eta})$

$$\frac{1}{2}\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \quad \frac{1}{3}\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix},$$

staircase   braided (simplified)

Comparison of Deterministic Codes and Ensembles

Deterministic



$$\boldsymbol{x}^{(\ell)} = \boldsymbol{\Psi}_{\geq t}(c\boldsymbol{B}\boldsymbol{x}^{(\ell-1)})$$

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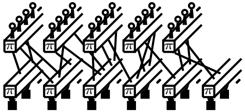
$$\frac{1}{2}\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \quad \frac{1}{3}\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix},$$

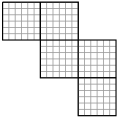
staircase | braided (simplified)

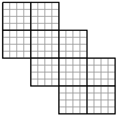
Ensemble-Based [Jian et al., 2012]



Comparison of Deterministic Codes and Ensembles

Deterministic





$$\boldsymbol{x}^{(\ell)} = \boldsymbol{\Psi}_{\geq t}(c\boldsymbol{B}\boldsymbol{x}^{(\ell-1)})$$

$(\boldsymbol{B} = \gamma\boldsymbol{\eta})$

$$\frac{1}{2}\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \quad \frac{1}{3}\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix},$$

staircase　　braided (simplified)

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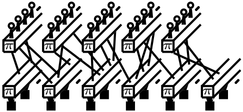
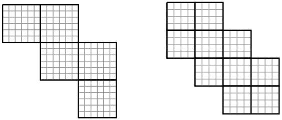


$$\boldsymbol{x}^{(\ell)} = \boldsymbol{\Psi}_{\geq t}(c\tilde{\boldsymbol{B}}\boldsymbol{x}^{(\ell-1)})$$

Comparison of Deterministic Codes and Ensembles

Deterministic



$$\boldsymbol{x}^{(\ell)} = \boldsymbol{\Psi}_{\geq t}(c\boldsymbol{B}\boldsymbol{x}^{(\ell-1)})$$

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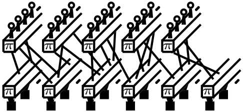
$$\frac{1}{2}\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \quad \frac{1}{3}\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix},$$

staircase — braided (simplified)

Ensemble-Based [Jian et al., 2012]

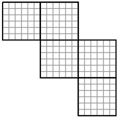


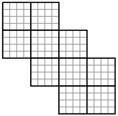
$$\boldsymbol{x}^{(\ell)} = \boldsymbol{\Psi}_{\geq t}(c\tilde{\boldsymbol{B}}\boldsymbol{x}^{(\ell-1)})$$

- Density evolution equations have the same form

Comparison of Deterministic Codes and Ensembles

Deterministic





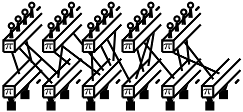
$$\boldsymbol{x}^{(\ell)} = \boldsymbol{\Psi}_{\geq t}(c\boldsymbol{B}\boldsymbol{x}^{(\ell-1)})$$

$$(\boldsymbol{B} = \gamma\boldsymbol{\eta})$$

$$\frac{1}{2}\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \quad \frac{1}{3}\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix},$$

staircase — braided (simplified)

Ensemble-Based [Jian et al., 2012]



$$\boldsymbol{x}^{(\ell)} = \boldsymbol{\Psi}_{\geq t}(c\tilde{\boldsymbol{B}}\boldsymbol{x}^{(\ell-1)})$$

$$(\tilde{\boldsymbol{B}} = \boldsymbol{A}^{\intercal}\boldsymbol{A})$$

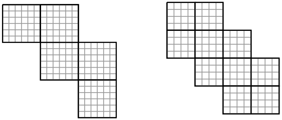
$$\boldsymbol{A} = \frac{1}{w}\begin{pmatrix} 1 & 1 & \cdots & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & \cdots & 1 & 0 & 0 \\ 0 & 0 & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & 0 & 1 & 1 & \cdots & 1 \end{pmatrix}$$

w (coupling width)

- Density evolution equations have the same form

Comparison of Deterministic Codes and Ensembles

Deterministic



$$x^{(\ell)} = \Psi_{\geq t}(cBx^{(\ell-1)})$$

$(B = \gamma\eta)$

$$\frac{1}{2}\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \quad \frac{1}{3}\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix},$$

staircase — braided (simplified)

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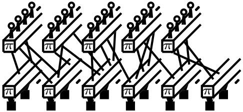


$$x^{(\ell)} = \Psi_{\geq t}(c\tilde{B}x^{(\ell-1)})$$

$(\tilde{B} = A^{\intercal}A)$

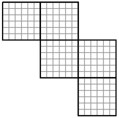
$$\frac{1}{4}\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}, \quad \frac{1}{9}\begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 2 & 2 & 1 & 0 & 0 \\ 1 & 2 & 3 & 2 & 1 & 0 \\ 0 & 1 & 2 & 3 & 2 & 1 \\ 0 & 0 & 1 & 2 & 2 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix}$$

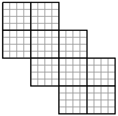
$w = 2$ — $w = 3$

- Density evolution equations have the same form

Comparison of Deterministic Codes and Ensembles

Deterministic





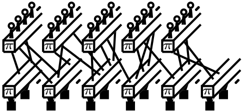
$$\boldsymbol{x}^{(\ell)} = \boldsymbol{\Psi}_{\geq t}(c\boldsymbol{B}\boldsymbol{x}^{(\ell-1)})$$

$(\boldsymbol{B} = \gamma\boldsymbol{\eta})$

$$\frac{1}{2}\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \quad \frac{1}{3}\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix},$$

staircase — braided (simplified)

Ensemble-Based [Jian et al., 2012]



capacity-achieving at high rates over the binary symmetric channel

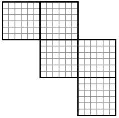
$$\frac{1}{4}\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}, \quad \frac{1}{9}\begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 2 & 2 & 1 & 0 & 0 \\ 1 & 2 & 3 & 2 & 1 & 0 \\ 0 & 1 & 2 & 3 & 2 & 1 \\ 0 & 0 & 1 & 2 & 2 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix}$$

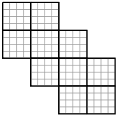
$w = 2$ — $w = 3$

- Density evolution equations have the same form

Comparison of Deterministic Codes and Ensembles

Deterministic

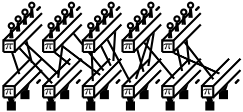




$$\boldsymbol{x}^{(\ell)} = \boldsymbol{\Psi}_{\geq t}(c\boldsymbol{B}\boldsymbol{x}^{(\ell-1)})$$

$$(\boldsymbol{B} = \gamma\boldsymbol{\eta})$$

Ensemble-Based [Jian et al., 2012]



capacity-achieving at high rates over the binary symmetric channel

$$\frac{1}{2}\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \quad \frac{1}{3}\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}, \quad \frac{1}{4}\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}, \quad \frac{1}{9}\begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 2 & 2 & 1 & 0 & 0 \\ 1 & 2 & 3 & 2 & 1 & 0 \\ 0 & 1 & 2 & 3 & 2 & 1 \\ 0 & 0 & 1 & 2 & 2 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix}$$

staircase | braided (simplified) | $w = 2$ | $w = 3$

- Density evolution equations have the same form
- Different averaging due to matrices $\boldsymbol{B}$ and $\tilde{\boldsymbol{B}}$

Ensemble Performance via Deterministic Codes

Ensemble Performance via Deterministic Codes

Can we "emulate" the ensemble density evolution with deterministic codes?

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Can we "emulate" the ensemble density evolution with deterministic codes?

1. Let $\boldsymbol{P} \triangleq w^2 \tilde{\boldsymbol{B}} = w^2 \boldsymbol{A}^\intercal \boldsymbol{A}$

Ensemble Performance via Deterministic Codes

Can we "emulate" the ensemble density evolution with deterministic codes?

1. Let $\boldsymbol{P} \triangleq w^2 \tilde{\boldsymbol{B}} = w^2 \boldsymbol{A}^\intercal \boldsymbol{A}$

$$\begin{array}{cccccc} 1 & 1 & & & & \\ 1 & 2 & 1 & & & \\ & 1 & 2 & 1 & & \\ & & 1 & 2 & 1 & \\ & & & 1 & 2 & 1 \\ & & & & 1 & 1 \end{array}$$

$\boldsymbol{P}$ for $L = 6$, $w = 2$

Ensemble Performance via Deterministic Codes

Can we "emulate" the ensemble density evolution with deterministic codes?

1. Let $\boldsymbol{P} \triangleq w^2 \tilde{\boldsymbol{B}} = w^2 \boldsymbol{A}^\intercal \boldsymbol{A}$
2. Form $\boldsymbol{\eta}'$ from $\boldsymbol{P}$: replace entries $P_{i,j}$ by symmetric $w \times w$ matrices with $P_{i,j}$ ones per row/column

$$
\begin{matrix}
1 & 1 & & & & \\
1 & 2 & 1 & & & \\
 & 1 & 2 & 1 & & \\
 & & 1 & 2 & 1 & \\
 & & & 1 & 2 & 1 \\
 & & & & 1 & 1
\end{matrix}
$$

$\boldsymbol{P}$ for $L = 6$, $w = 2$

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$$
\begin{array}{cccccc}
\begin{smallmatrix}1 & 0\\ 0 & 1\end{smallmatrix} & 1 & & & & \\
1 & 2 & 1 & & & \\
 & 1 & 2 & 1 & & \\
 & & 1 & 2 & 1 & \\
 & & & 1 & 2 & 1 \\
 & & & & 1 & 1
\end{array}
$$

$\boldsymbol{P}$ for $L = 6$, $w = 2$

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2. Form $\boldsymbol{\eta}'$ from $\boldsymbol{P}$: replace entries $P_{i,j}$ by symmetric $w \times w$ matrices with $P_{i,j}$ ones per row/column

$$
\begin{array}{cccccc}
\begin{smallmatrix}1 & 0\\ 0 & 1\end{smallmatrix} & 1 & & & & \\
1 & \begin{smallmatrix}1 & 1\\ 1 & 1\end{smallmatrix} & 1 & & & \\
 & 1 & 2 & 1 & & \\
 & & 1 & 2 & 1 & \\
 & & & 1 & 2 & 1 \\
 & & & & 1 & 1
\end{array}
$$

$\boldsymbol{P}$ for $L = 6$, $w = 2$

Ensemble Performance via Deterministic Codes

Can we "emulate" the ensemble density evolution with deterministic codes?

1. Let $\boldsymbol{P} \triangleq w^2 \tilde{\boldsymbol{B}} = w^2 \boldsymbol{A}^\intercal \boldsymbol{A}$
2. Form $\boldsymbol{\eta}'$ from $\boldsymbol{P}$: replace entries $P_{i,j}$ by symmetric $w \times w$ matrices with $P_{i,j}$ ones per row/column

$$
\begin{array}{cccccccccccc}
1 & 0 & 1 & 0 & & & & & & & & \\
0 & 1 & 0 & 1 & & & & & & & & \\
1 & 0 & 1 & 1 & 1 & 0 & & & & & & \\
0 & 1 & 1 & 1 & 0 & 1 & & & & & & \\
 & & 1 & 0 & 1 & 1 & 1 & 0 & & & & \\
 & & 0 & 1 & 1 & 1 & 0 & 1 & & & & \\
 & & & & 1 & 0 & 1 & 1 & 1 & 0 & & \\
 & & & & 0 & 1 & 1 & 1 & 0 & 1 & & \\
 & & & & & & 1 & 0 & 1 & 1 & 1 & 0 \\
 & & & & & & 0 & 1 & 1 & 1 & 0 & 1 \\
 & & & & & & & & 1 & 0 & 1 & 0 \\
 & & & & & & & & 0 & 1 & 0 & 1
\end{array}
$$

$\boldsymbol{\eta}'$ (not unique)

Ensemble Performance via Deterministic Codes

Can we "emulate" the ensemble density evolution with deterministic codes?

1. Let $\boldsymbol{P} \triangleq w^2 \tilde{\boldsymbol{B}} = w^2 \boldsymbol{A}^\intercal \boldsymbol{A}$
2. Form $\boldsymbol{\eta}'$ from $\boldsymbol{P}$: replace entries $P_{i,j}$ by symmetric $w \times w$ matrices with $P_{i,j}$ ones per row/column
3. $\eta_{2i,2j-1} = \eta'_{i,j}$ and $\eta_{2i-1,2j} = \eta'_{j,i}$ (create row/column array structure)

$$
\begin{array}{cccccccccccc}
1 & 0 & 1 & 0 & & & & & & & & \\
0 & 1 & 0 & 1 & & & & & & & & \\
1 & 0 & 1 & 1 & 1 & 0 & & & & & & \\
0 & 1 & 1 & 1 & 0 & 1 & & & & & & \\
 & & 1 & 0 & 1 & 1 & 1 & 0 & & & & \\
 & & 0 & 1 & 1 & 1 & 0 & 1 & & & & \\
 & & & & 1 & 0 & 1 & 1 & 1 & 0 & & \\
 & & & & 0 & 1 & 1 & 1 & 0 & 1 & & \\
 & & & & & & 1 & 0 & 1 & 1 & 1 & 0 \\
 & & & & & & 0 & 1 & 1 & 1 & 0 & 1 \\
 & & & & & & & & 1 & 0 & 1 & 0 \\
 & & & & & & & & 0 & 1 & 0 & 1
\end{array}
$$

$\boldsymbol{\eta}'$ (not unique)

Ensemble Performance via Deterministic Codes

Can we "emulate" the ensemble density evolution with deterministic codes?

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2. Form $\boldsymbol{\eta}'$ from $\boldsymbol{P}$: replace entries $P_{i,j}$ by symmetric $w \times w$ matrices with $P_{i,j}$ ones per row/column
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$$
\begin{array}{cccccccccccc}
1 & 0 & 1 & 0 & & & & & & & & \\
0 & 1 & 0 & 1 & & & & & & & & \\
1 & 0 & 1 & 1 & 1 & 0 & & & & & & \\
0 & 1 & 1 & 1 & 0 & 1 & & & & & & \\
 & & 1 & 0 & 1 & 1 & 1 & 0 & & & & \\
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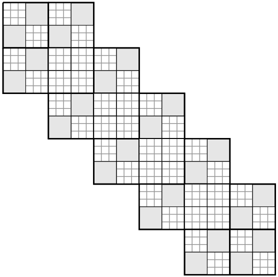

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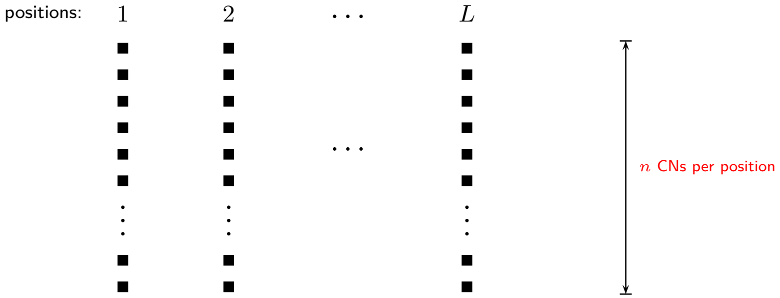

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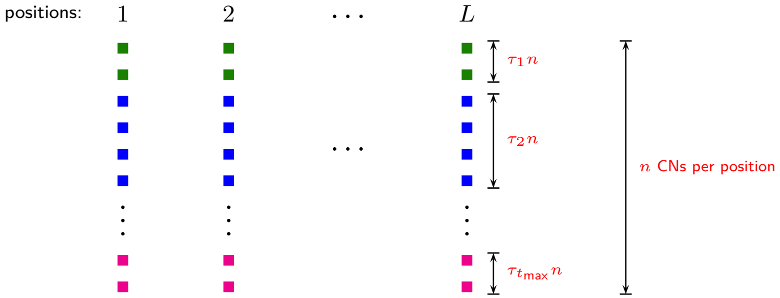

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$\Longrightarrow$ no asymptotic ($n \to \infty$, $\ell \to \infty$, large w) performance improvement possible

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Thank you!

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