Feldman, Karger, Wainwright - LP Decoding (2003)

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Gothenburg, Sweden

May 21, 2012

CHALMERS

Title Translation

LP Decoding

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Linear Programming Decoding

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Linear Programming for finding the mode of a posterior distribution in an NP-hard combinatorial inference problem

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Linear Programming for finding the mode of a posterior distribution in an NP-hard combinatorial inference problem

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A new approach and approximate solutions for finding the mode of a posterior distribution in an NP-hard combinatorial inference problem

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Introduction

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Channel Coding

It is possible to provide (almost) error-free performance, even in the presence of bit-flipps.

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- It turns out that for our channel (and equiprobable codewords)

$$\hat{\boldsymbol{x}} = \arg\min_{\boldsymbol{x} \in \mathcal{C}} \mathrm{d_H}(\boldsymbol{x}, \boldsymbol{y}),$$

where $\mathrm{d_H}$ is the Hamming distance.

Graphical Interpretation

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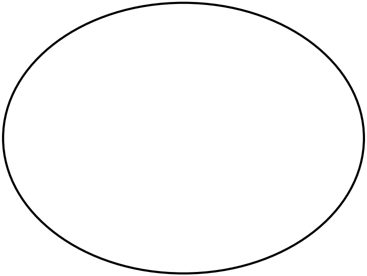

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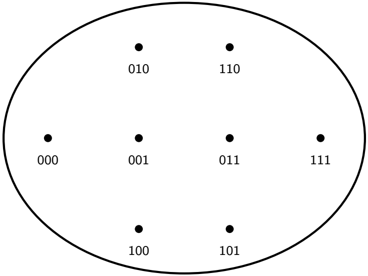

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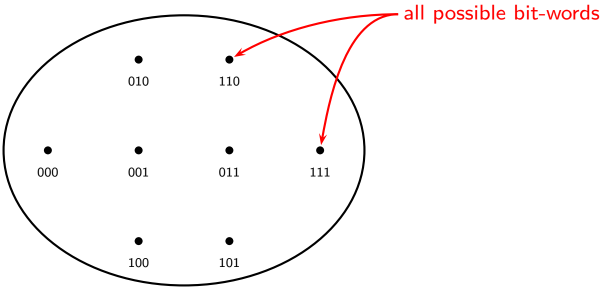

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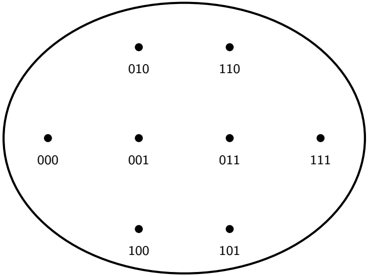

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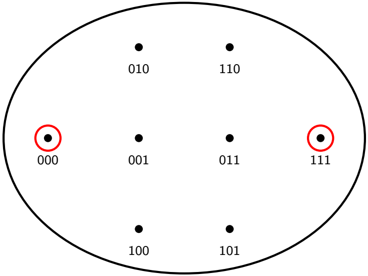

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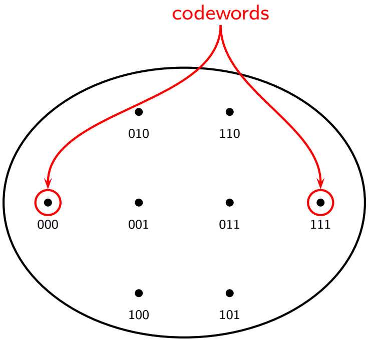

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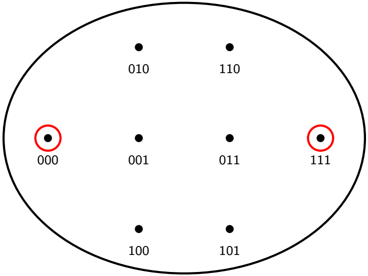

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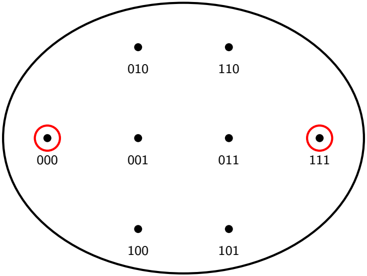

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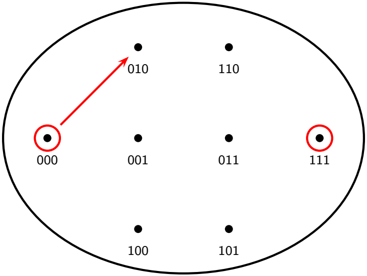

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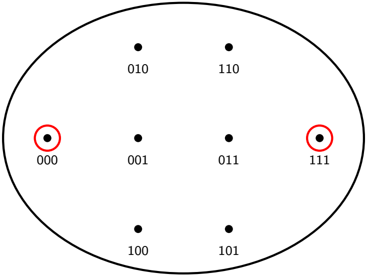

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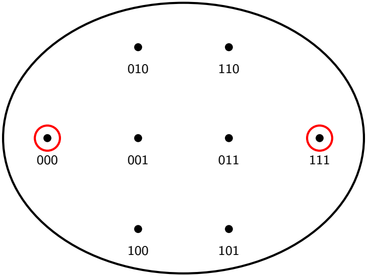

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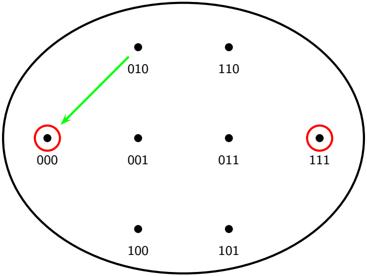

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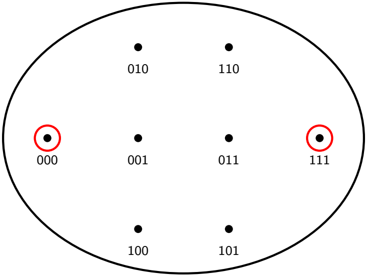

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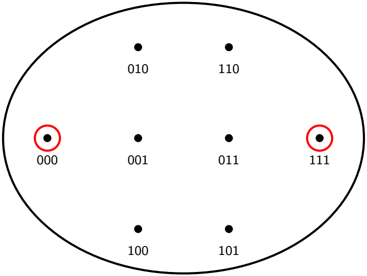

Fundamental Limitation

An error cannot be corrected if it is so large that one moves too far away from the true codeword and is now, in fact, closer to another codeword.

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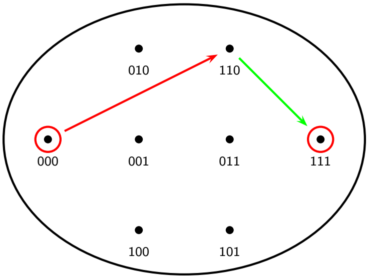

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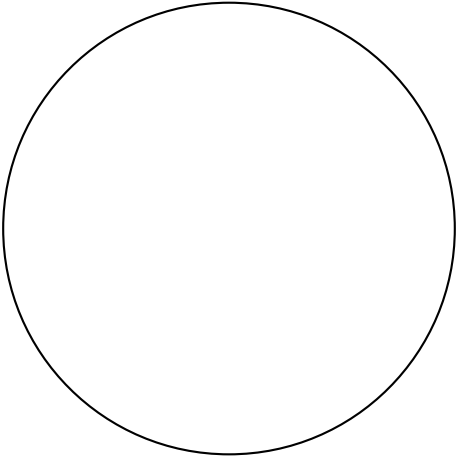

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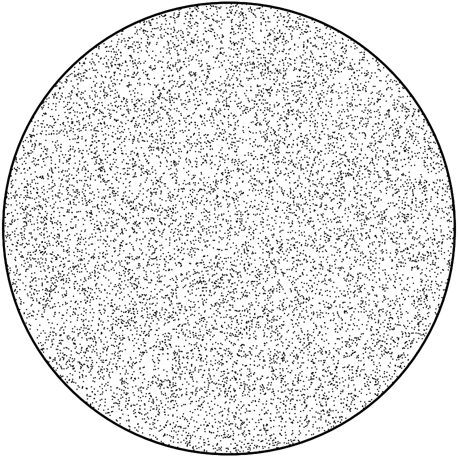

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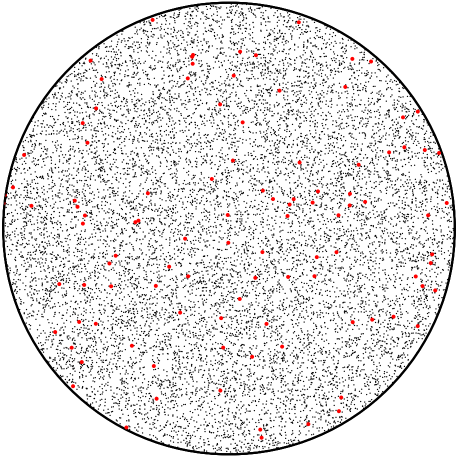

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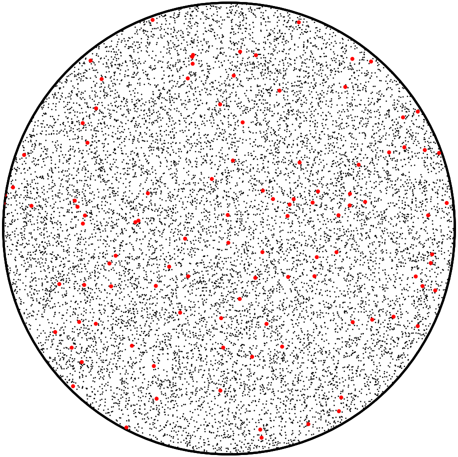

Optimally solving the inference problem becomes impractical for long codes.

A (Very) Short Introduction to Linear Programming

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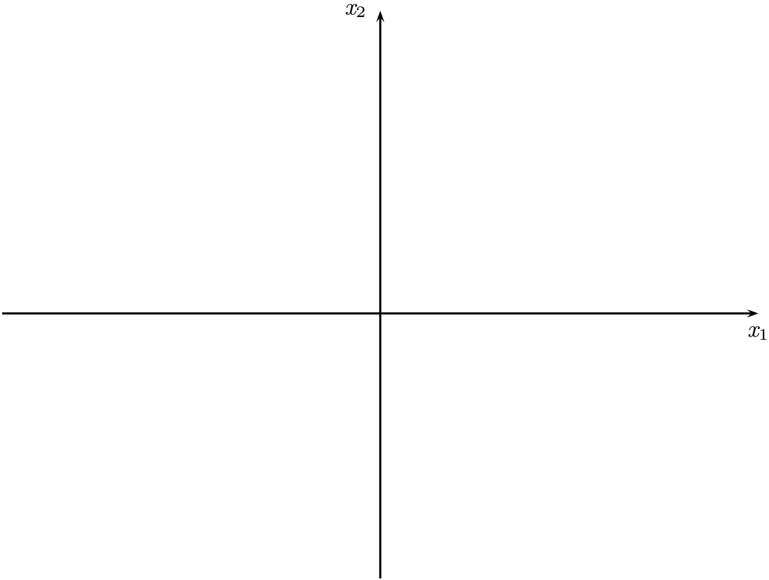

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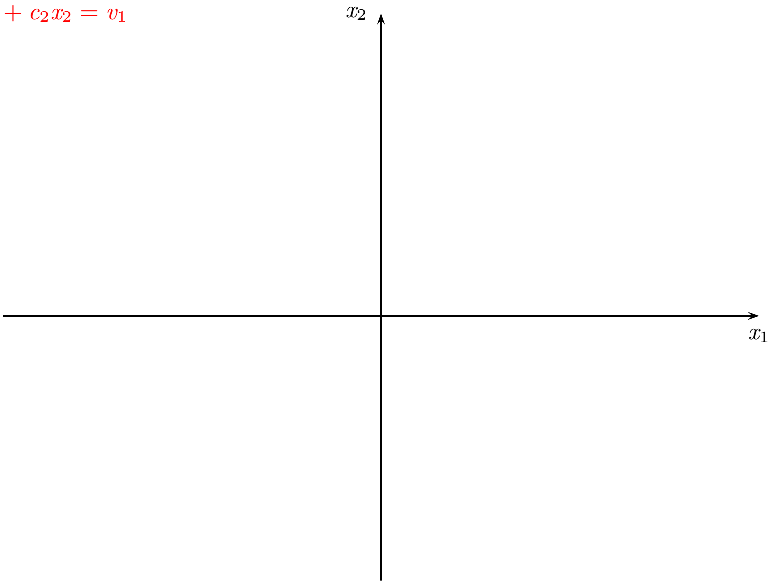

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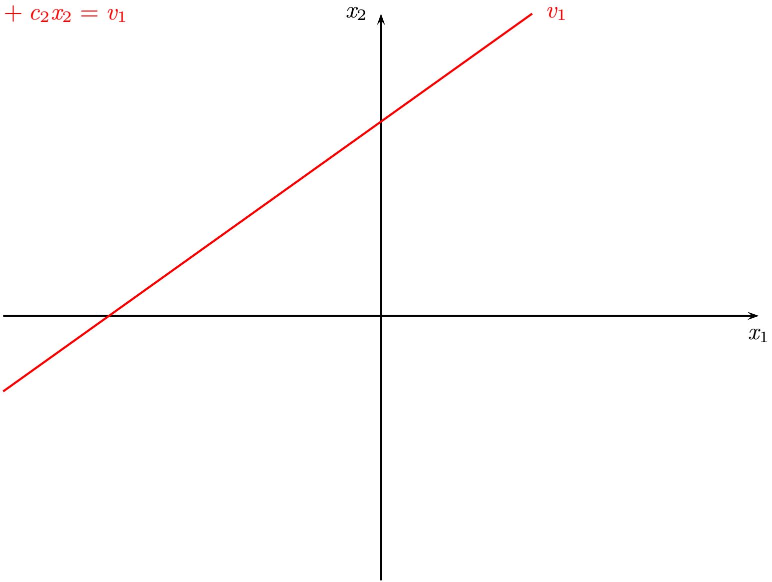

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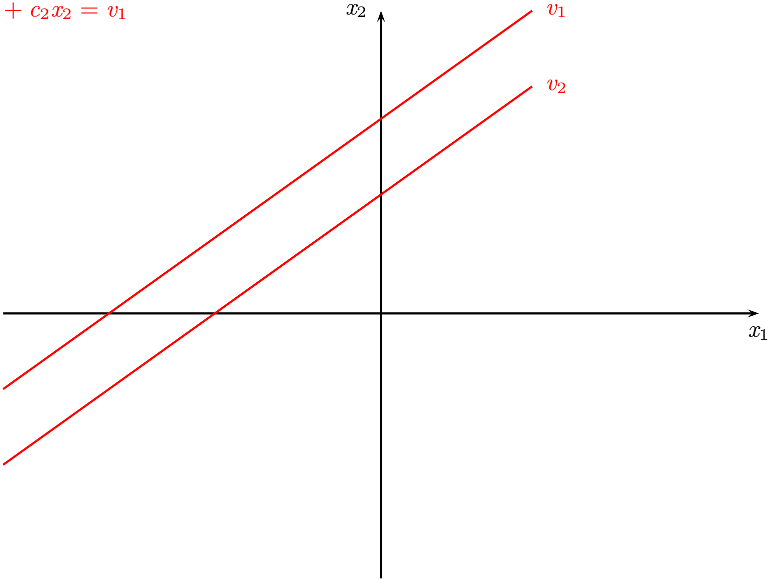

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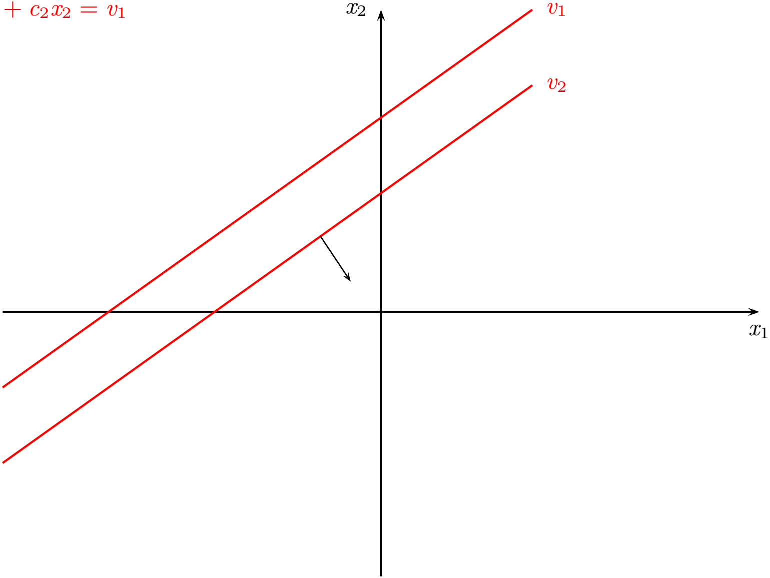

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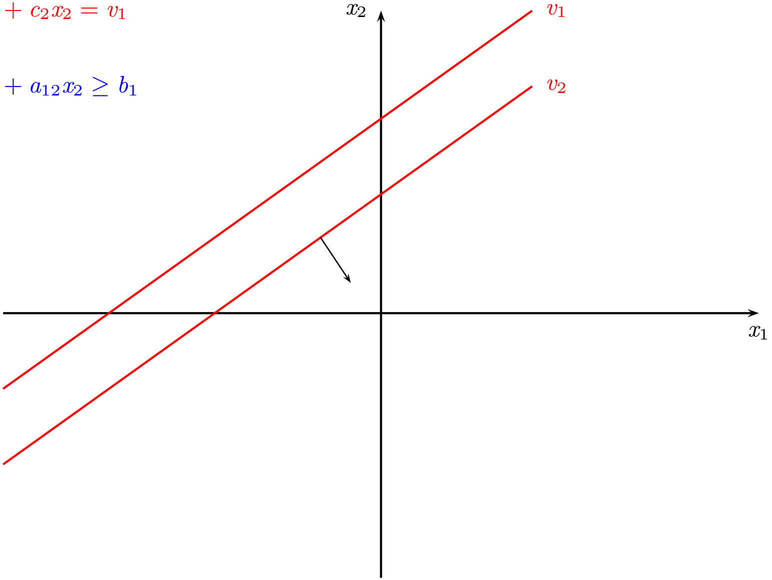

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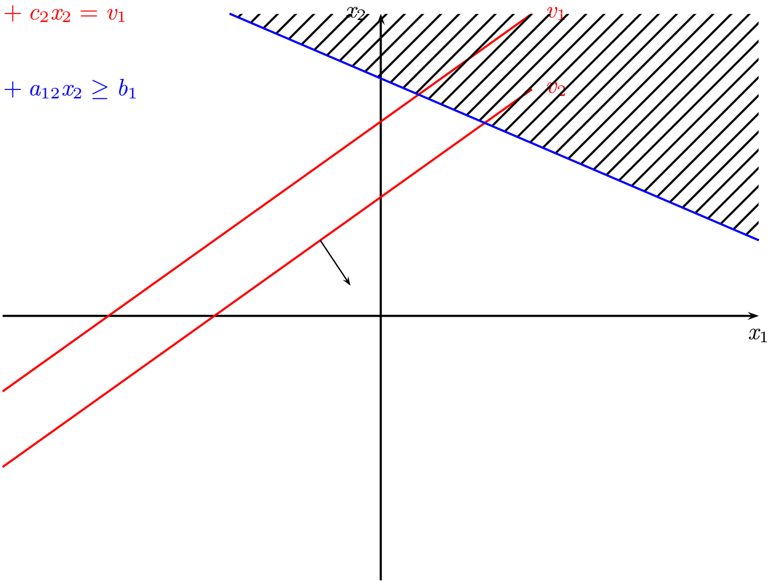

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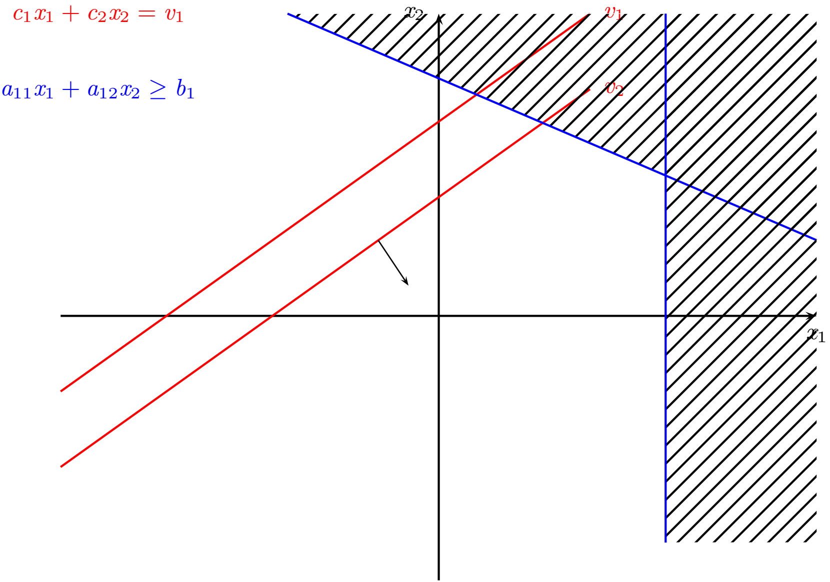

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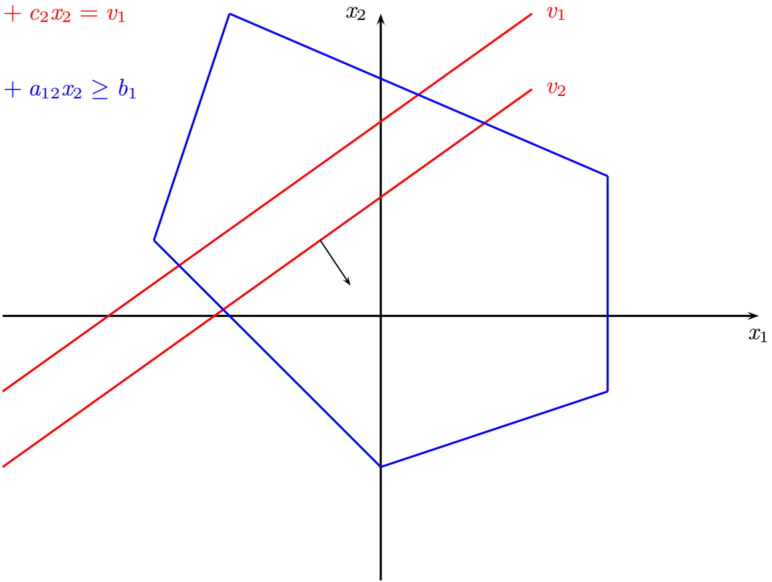

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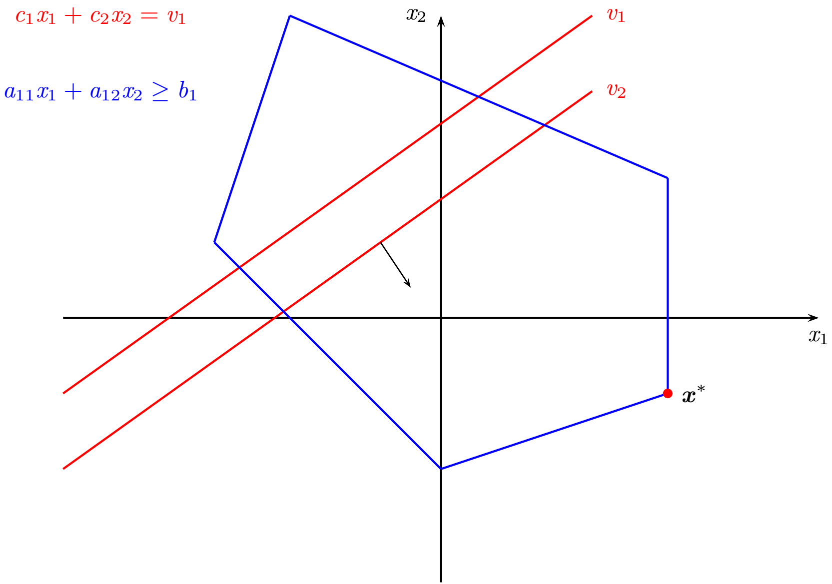

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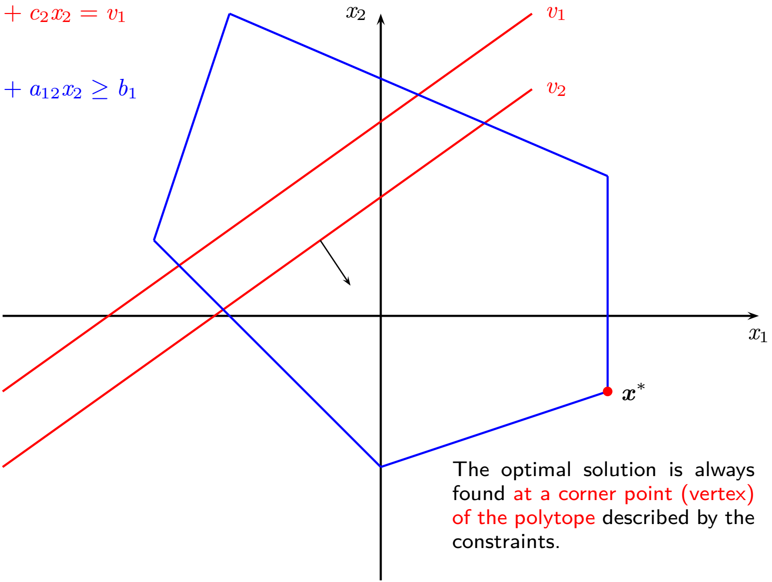

The optimal solution is always found at a corner point (vertex) of the polytope described by the constraints.

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Main Idea

Formulation of the decoding problem as a linear program with efficient polytopes (in the number of constraints).

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- Replace the optimization over $\mathcal{C}$ by an optimization over a relaxed polytope $\mathcal{P} \subseteq [0,1]^n$:

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- LP Decoding: Solve (1). Output $\hat{\boldsymbol{x}}_{\mathsf{LP}}$ if the solution is integral ($\in \{0,1\}^n$), otherwise output "error".

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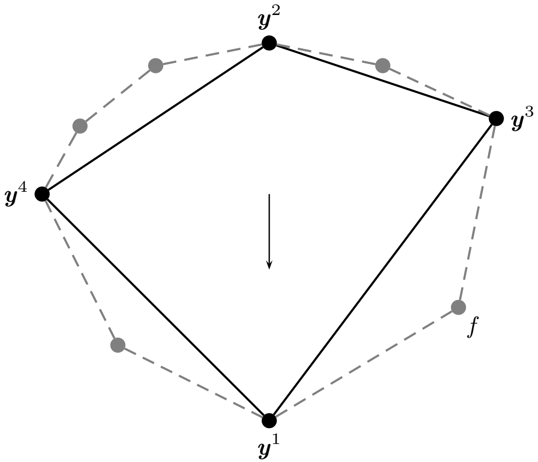
- Remarkable property for a suboptimal decoder
- Choosing $\mathcal{P}$ as the convex hull of $\mathcal{C}$ recovers the optimal decoder, but is again impractical.

Geometric Perspective



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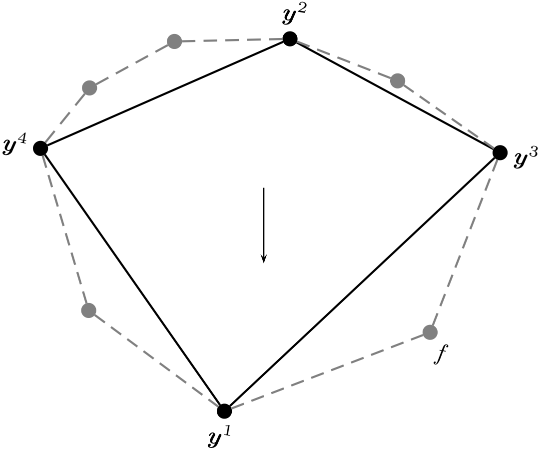

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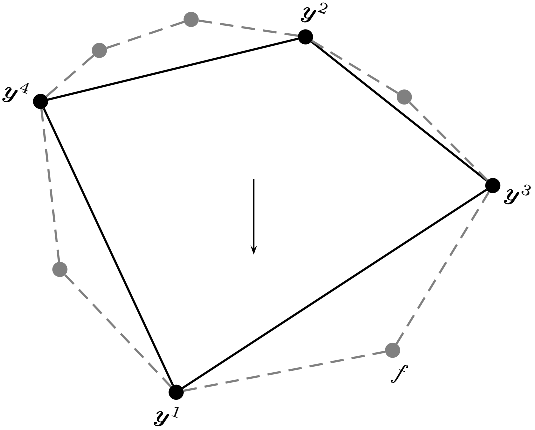

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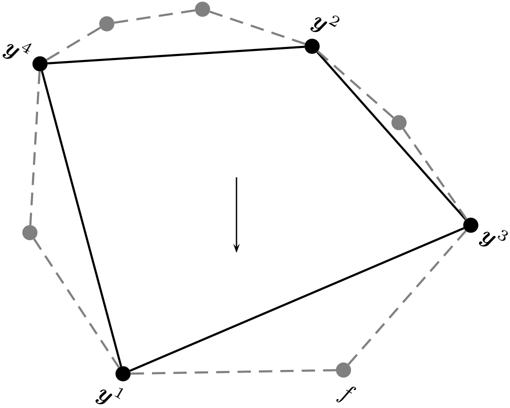

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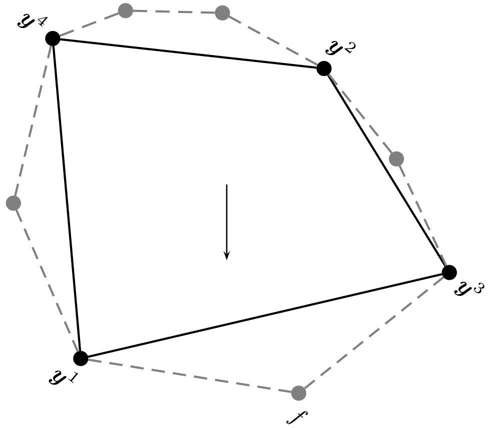

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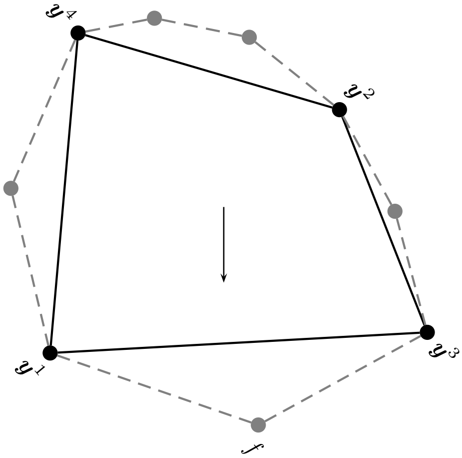

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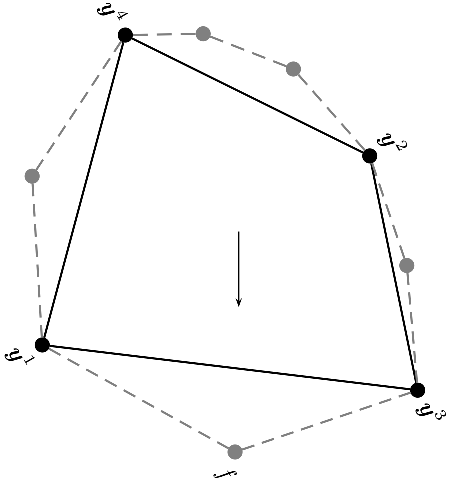

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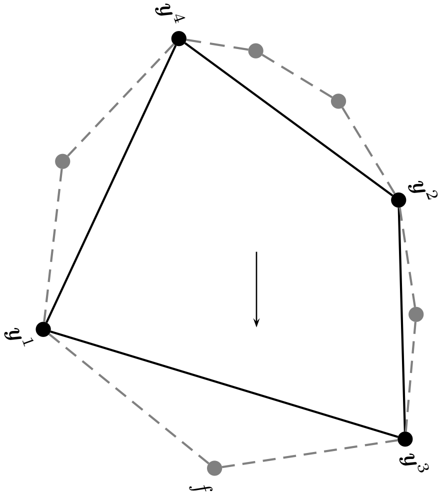

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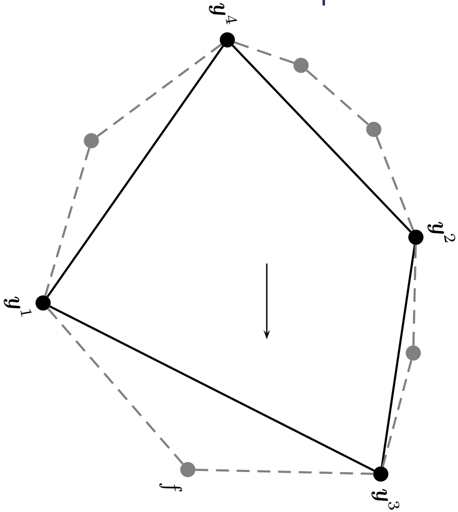

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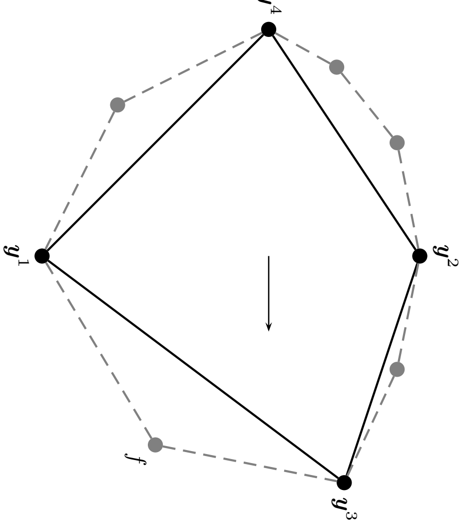

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entity	*property*	*significance*
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entity	*property*	*consequence*
binary code $\mathcal{C}$	is linear	error prob. does not depend on $\boldsymbol{x}$
polytope $\mathcal{P}$	is $\mathcal{C}$-symmetric	error prob. does not depend on $\boldsymbol{x}$

Comparison of Suboptimal Decoding Algorithms for LDPC Codes

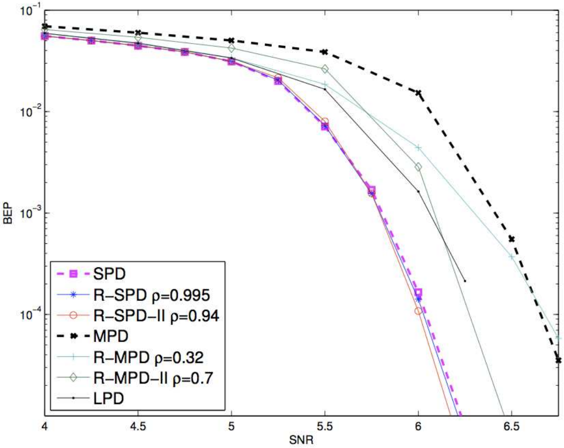

Figure 6. BEP after 20 iterations as a function of the SNR, with optimized values of ρ.

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- not dispair when someone tells you that your problem is NP-hard.