Coding and Deep Learning for High-Speed Fiber-Optic Communication Systems

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Thanks to: Henry Pfister[2], Alexandre Graell i Amat[1], Fredrik Brännström[1], and Erik Agrell[1]

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TU Munich
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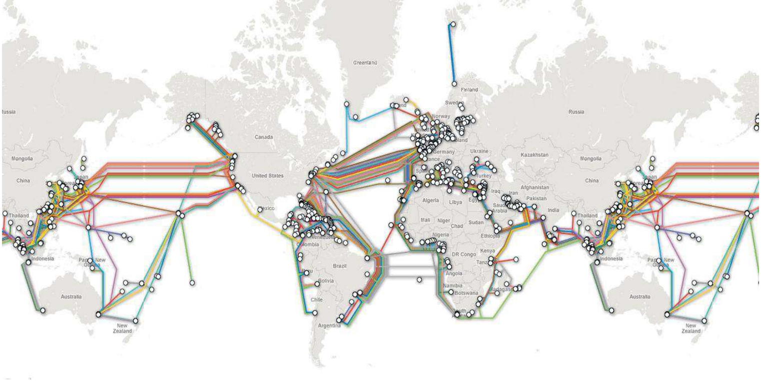
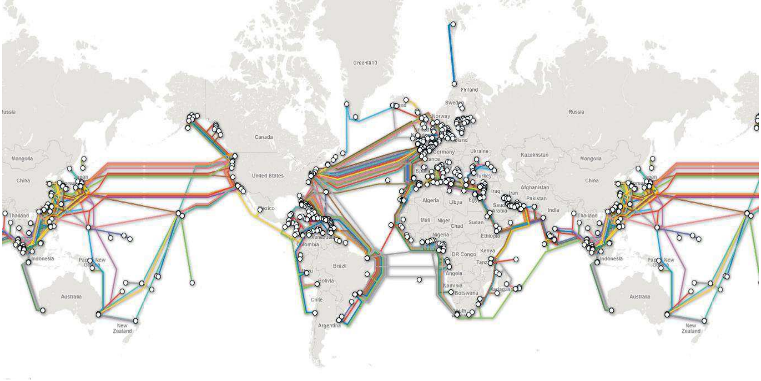
Fiber-Optic Communications



Fiber-Optic Communications



Fiber-optic communication systems enable data traffic over very long distances connecting cities, countries, and continents.

Fiber-Optic Communications

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- Long distances result in significant signal attenuation

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- Periodic amplification necessary $\Longrightarrow$ random distortions or noise

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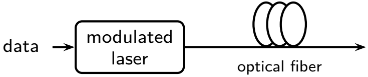

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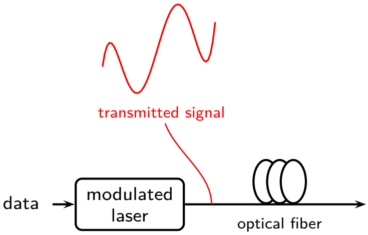

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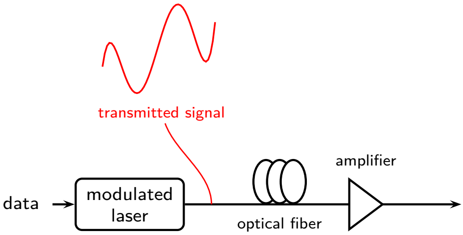

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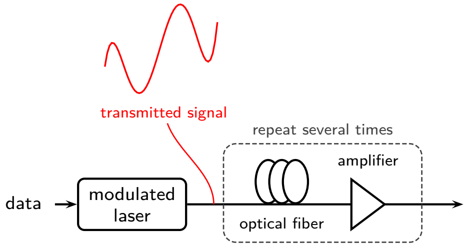

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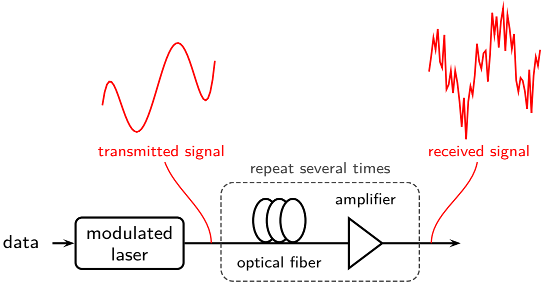

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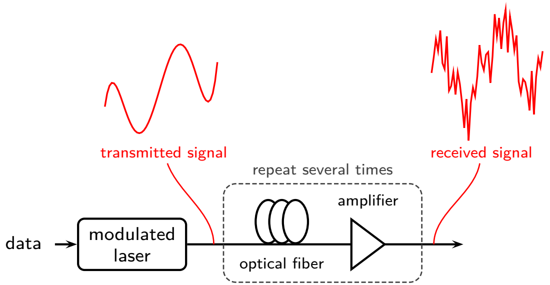

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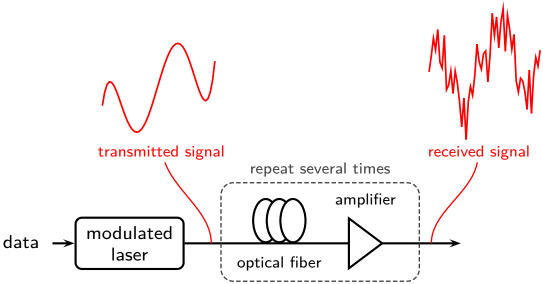
Outline

Part 1: Error-correcting codes to ensure reliable data transmission.

Fiber-Optic Communications



- Long distances result in significant signal attenuation
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- Fiber dispersion and nonlinearity $\implies$ deterministic distortions

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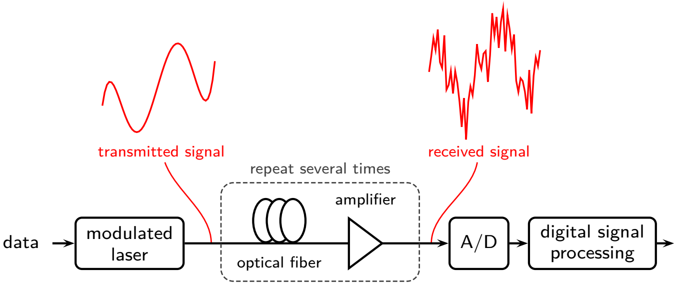

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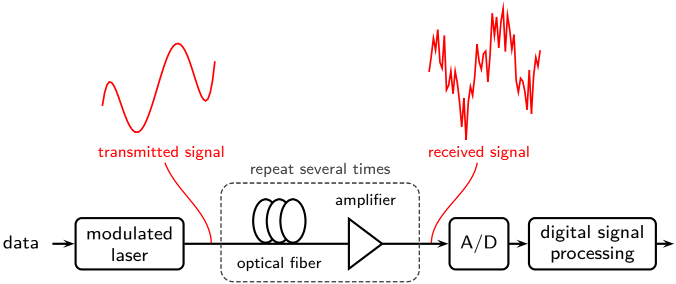

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- Periodic amplification necessary $\Longrightarrow$ random distortions or noise
- Fiber dispersion and nonlinearity $\Longrightarrow$ deterministic distortions

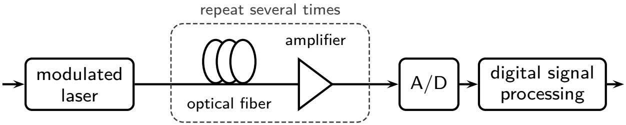
Outline

Part 1: Error-correcting codes to ensure reliable data transmission.

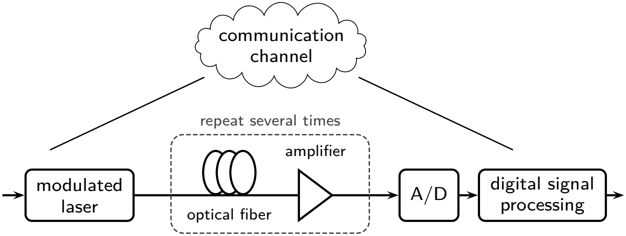
Part 2: Nonlinear equalization via deep learning tools.

Part 1: Coding

Error-Correcting Codes



Error-Correcting Codes



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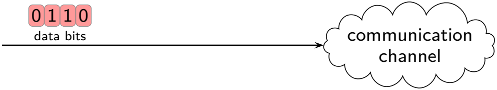

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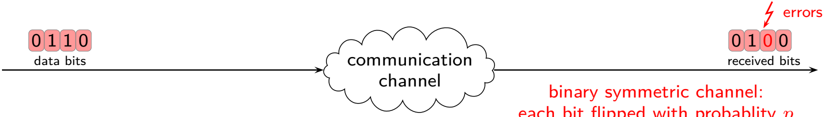
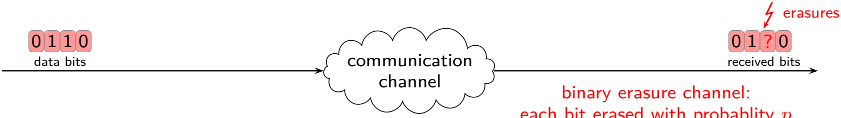


binary symmetric channel:
each bit flipped with probablity p

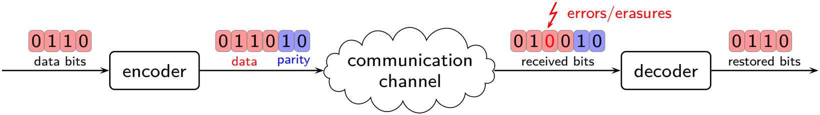
(Motivation: limited soft-information in metro networks, outer clean-up codes, . . .)

Error-Correcting Codes



binary erasure channel:
each bit erased with probablity p

Error-Correcting Codes



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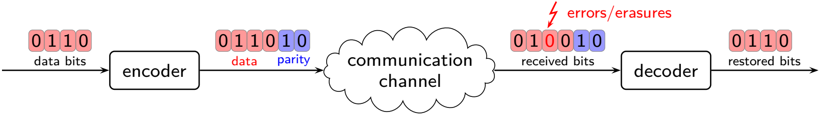
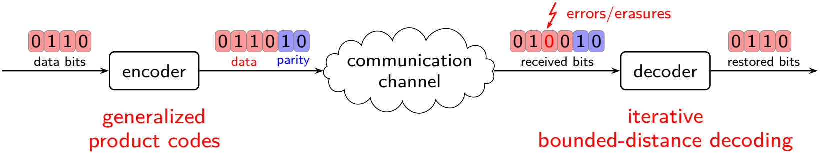


Requirements for Fiber-Optic Communications

- Very high throughputs (100 Gigabits per second or higher)
- Very high net coding gains (close-to-capacity performance)
- Very low bit error rates (below 10^{-15})

Error-Correcting Codes



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- Very high throughputs (100 Gigabits per second or higher)
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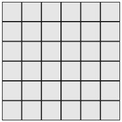
Outline: Part 1 (Coding)

1. Asymptotic performance of deterministic generalized product codes
2. Binary erasure channel vs. binary symmetric channel

Product Codes and Staircase Codes

Product Codes and Staircase Codes

rectangular array [Elias, 1954]



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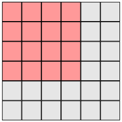

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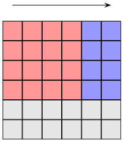

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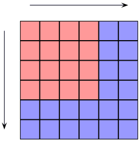

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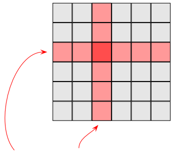

each row/column is a codeword in some component code

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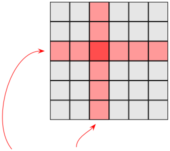

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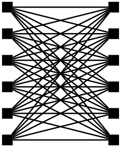
Tanner graph



constraint node degree = component code length

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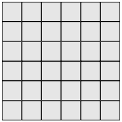

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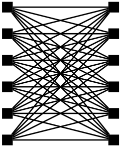

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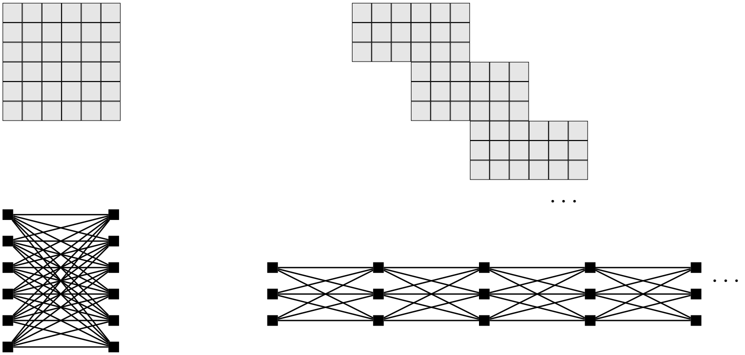
rectangular array [Elias, 1954]

staircase array [Smith et al., 2012]



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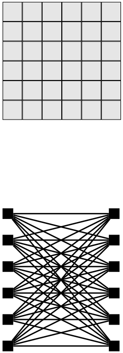

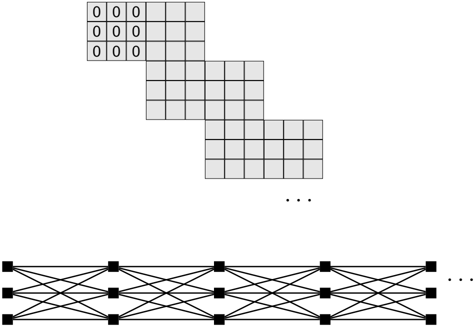

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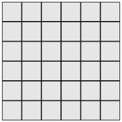

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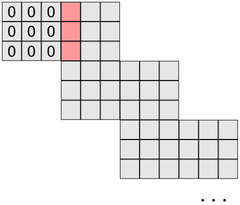

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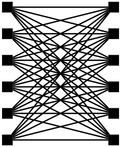
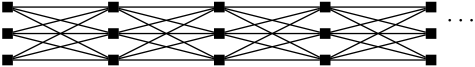
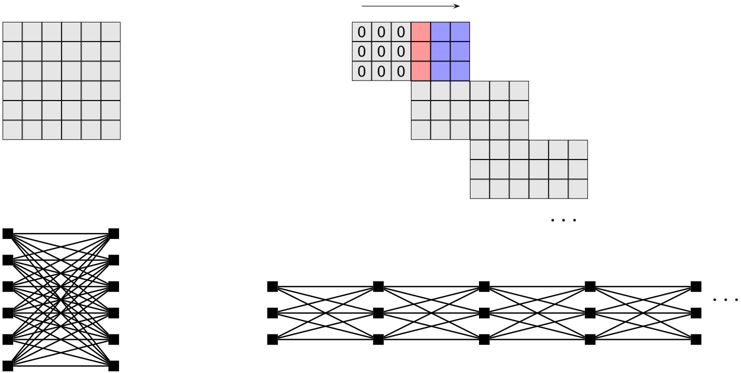




Product Codes and Staircase Codes

rectangular array [Elias, 1954] staircase array [Smith et al., 2012]

Tanner graph



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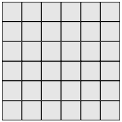

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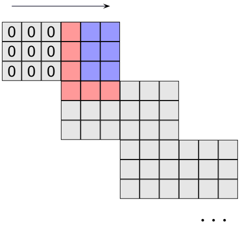

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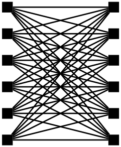

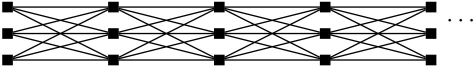

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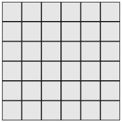

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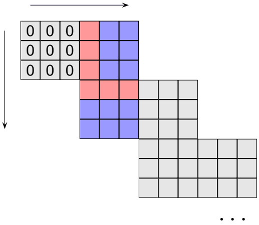

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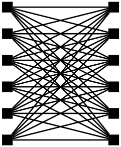

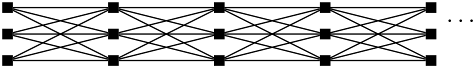

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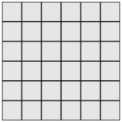

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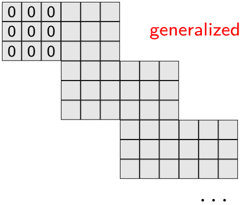
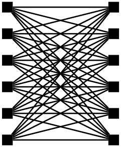
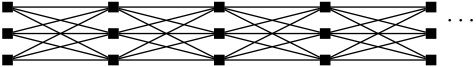


generalized product code (GPC)

Tanner graph





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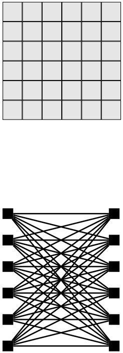
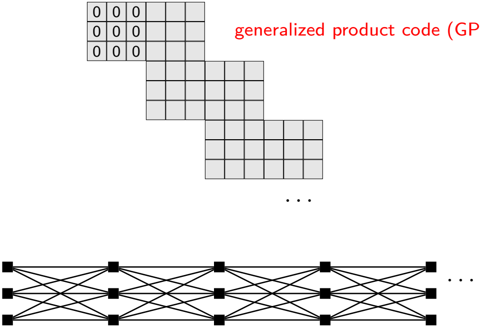
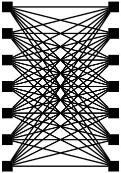


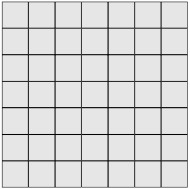


generalized product code (GPC)

- Deterministic codes with fixed and structured Tanner graph

Iterative Bounded-Distance Decoding





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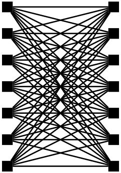
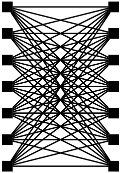


0	1	0	1	0	1	0
0	1	0	1	1	0	1
0	1	0	1	0	1	0
1	1	1	0	1	1	0
0	0	1	0	0	0	1
1	0	0	0	1	1	1
0	1	0	0	0	1	1

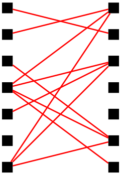
Iterative Bounded-Distance Decoding



0	?	0	?	0	1	?
?	1	0	1	1	0	1
0	1	0	?	0	?	?
1	1	1	0	1	1	0
0	0	1	0	0	0	1
1	0	?	?	1	1	?
0	1	0	?	0	1	1

- Codeword transmission over binary erasure channel with erasure probability p

Iterative Bounded-Distance Decoding



residual graph

0	?	0	?	0	1	?
?	1	0	1	1	0	1
0	1	0	?	0	?	?
1	1	1	0	1	1	0
0	0	1	0	0	0	1
1	0	?	?	1	1	?
0	1	0	?	0	1	1

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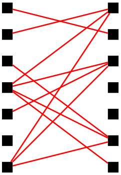

residual graph

0	?	0	?	0	1	?
?	1	0	1	1	0	1
0	1	0	?	0	?	?
1	1	1	0	1	1	0
0	0	1	0	0	0	1
1	0	?	?	1	1	?
0	1	0	?	0	1	1

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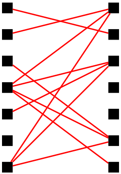
- Each component code corrects $\leq t$ erasures

Iterative Bounded-Distance Decoding



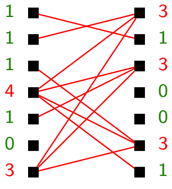
residual graph

0	?	0	?	0	1	?
?	1	0	1	1	0	1
0	1	0	?	0	?	?
1	1	1	0	1	1	0
0	0	1	0	0	0	1
1	0	?	?	1	1	?
0	1	0	?	0	1	1

- Codeword transmission over binary erasure channel with erasure probability p
- Each component code corrects $\leq t$ erasures
- ℓ iterations of bounded-distance decoding $=$ peeling of vertices with degree $\leq t$ (in parallel)

Iterative Bounded-Distance Decoding

1st iteration $(t = 2)$



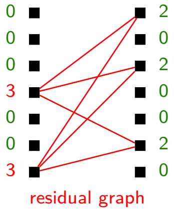
residual graph

0	?	0	?	0	1	?
?	1	0	1	1	0	1
0	1	0	?	0	?	?
1	1	1	0	1	1	0
0	0	1	0	0	0	1
1	0	?	?	1	1	?
0	1	0	?	0	1	1

- Codeword transmission over binary erasure channel with erasure probability p
- Each component code corrects $\leq t$ erasures
- ℓ iterations of bounded-distance decoding $=$ peeling of vertices with degree $\leq t$ (in parallel)

Iterative Bounded-Distance Decoding

2nd iteration $(t = 2)$



0	1	0	?	0	1	?
0	1	0	1	1	0	1
0	1	0	?	0	1	?
1	1	1	0	1	1	0
0	0	1	0	0	0	1
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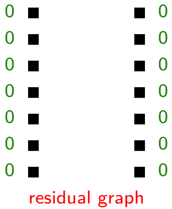

0	1	0	1	0	1	0
0	1	0	1	1	0	1
0	1	0	1	0	1	0
1	1	1	0	1	1	0
0	0	1	0	0	0	1
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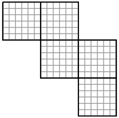
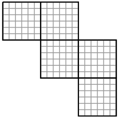
Performance Prediction

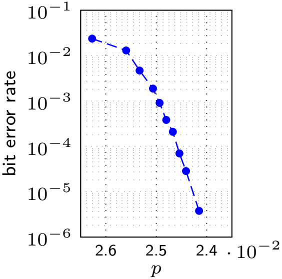


- Example: staircase code with a fixed component code

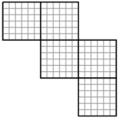
Performance Prediction

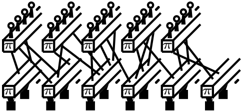


- Example: staircase code with a fixed component code
- Use simulations to predict performance → computationally intensive



Performance Prediction





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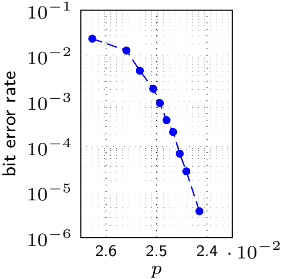
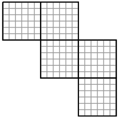
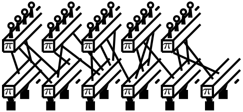


Performance Prediction





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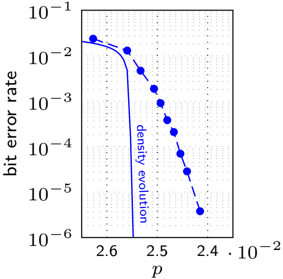
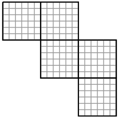
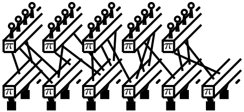


Performance Prediction





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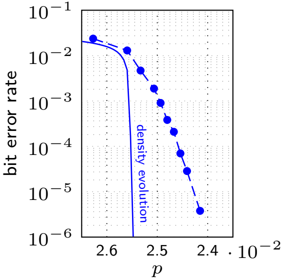
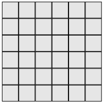


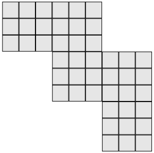
Is it possible to directly analyze deterministic generalized product codes?

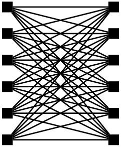
Density Evolution for Deterministic Generalized Product Codes

product codes

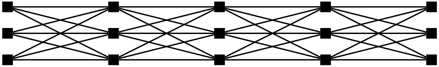


staircase codes





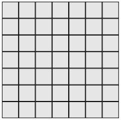
n: “problem size”, proportional to the component code length



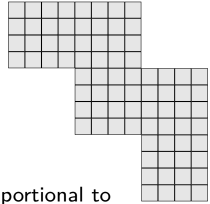
increasing n

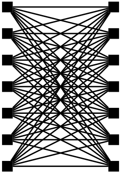
Density Evolution for Deterministic Generalized Product Codes

product codes



staircase codes





n: “problem size”, proportional to the component code length

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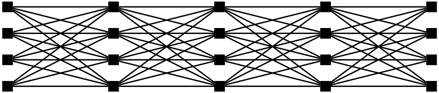

Density Evolution for Deterministic Generalized Product Codes

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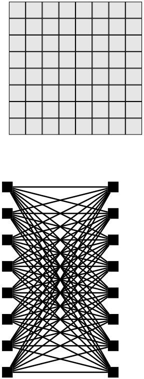

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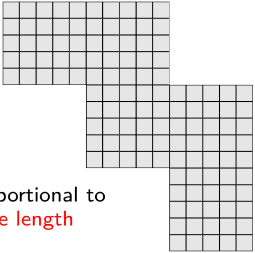

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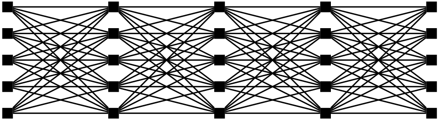
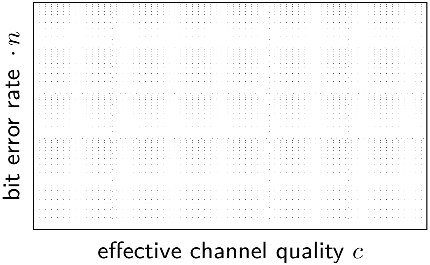


increasing n

Density Evolution

Density Evolution

- Let $p = c/n$ for $c > 0$, where c is the effective channel quality



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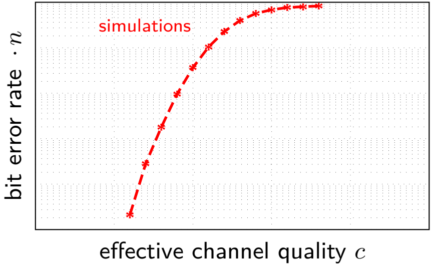

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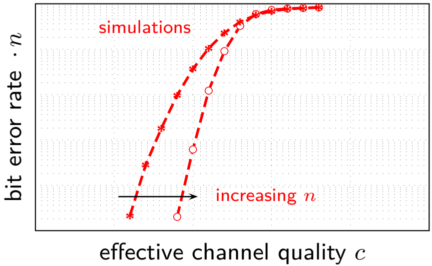

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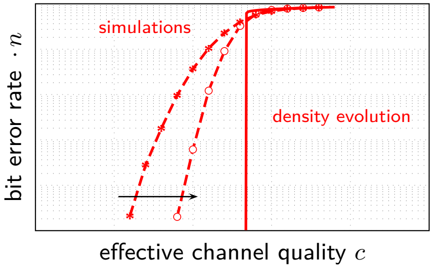

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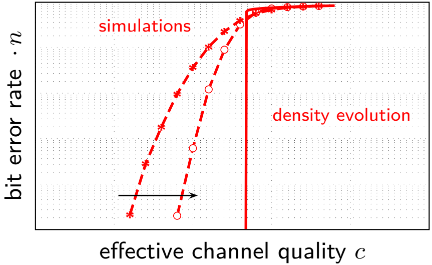
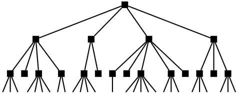


Proof and details in [Häger et al., 2017].
Key: convergence results for sparse random graphs [Bollobás et al., 2007]



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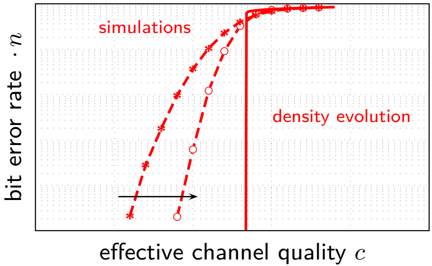

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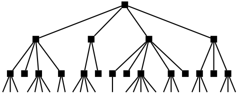

- Generalizes [Schwartz et al., 2005], [Justesen and Høholdt, 2007] to a large class of deterministic codes (staircase, braided, etc.); also works for different decoding schedules (e.g., window decoding)
- Applications: (asymptotic) performance prediction, code comparison via thresholds, efficient parameter optimization, . . .

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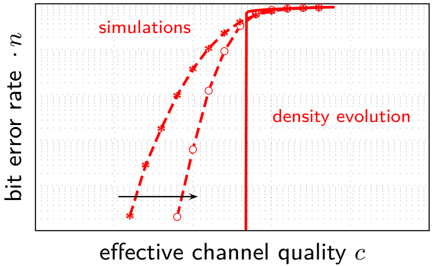

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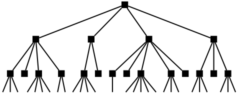

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only one small problem . . .

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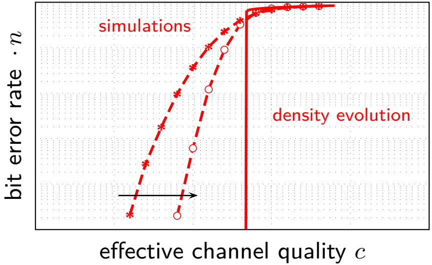

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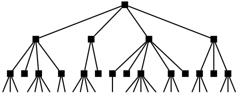

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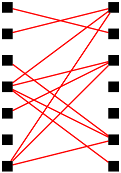
only one small problem . . . binary erasure channel is not the target channel

Binary Symmetric Channel vs. Binary Erasure Channel



residual graph

0	?	0	?	0	1	?
?	1	0	1	1	0	1
0	1	0	?	0	?	?
1	1	1	0	1	1	0
0	0	1	0	0	0	1
1	0	?	?	1	1	?
0	1	0	?	0	1	1

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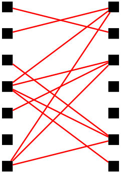

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0	0	0	0	0	1	1
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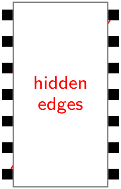

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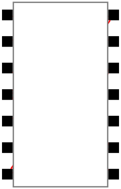

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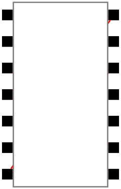

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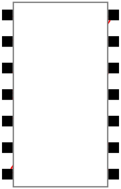
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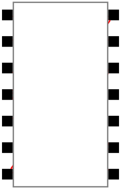

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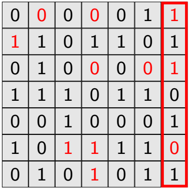

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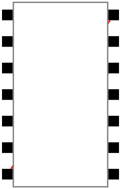

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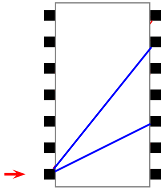

0	0	0	0	0	1	1
1	1	0	1	1	0	0
0	1	0	0	0	0	1
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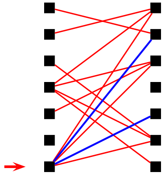

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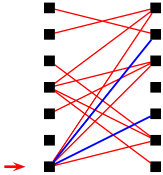
Binary Symmetric Channel vs. Binary Erasure Channel



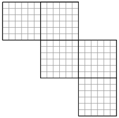
residual graph

0	0	0	0	0	1	1
1	1	0	1	1	0	0
0	1	0	0	0	0	1
1	1	1	0	1	1	0
0	0	1	0	0	0	0
1	0	1	1	1	1	0
0	1	0	1	0	1	1

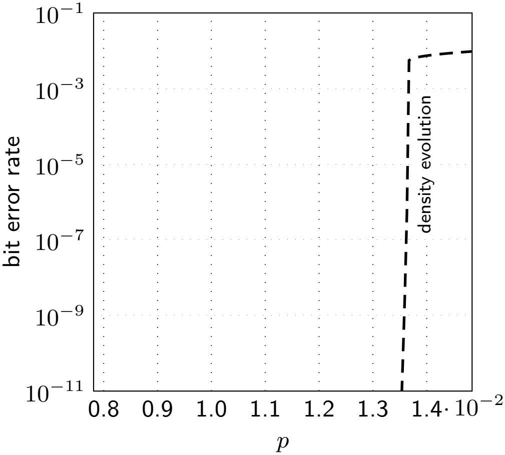
- Each component code corrects $\leq t$ errors
- Undetected errors during component decoding $\implies$ miscorrections
- Additional errors during iterative decoding

Performance Loss

Performance Loss

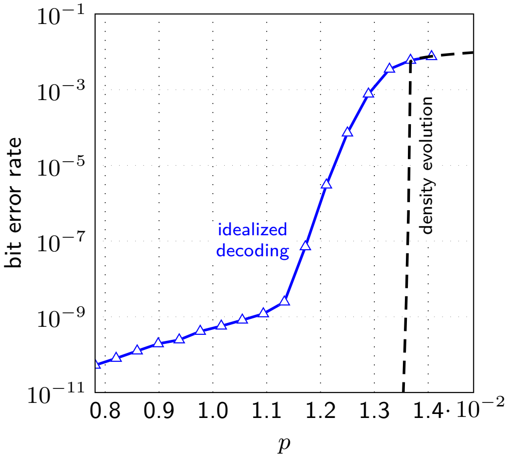


- Staircase code with $n = 256$ and $t = 2$

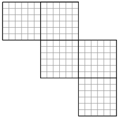


Performance Loss

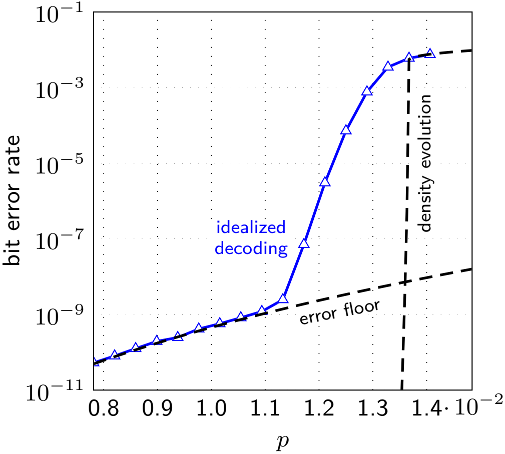
- Staircase code with $n = 256$ and $t = 2$



Performance Loss

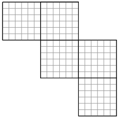


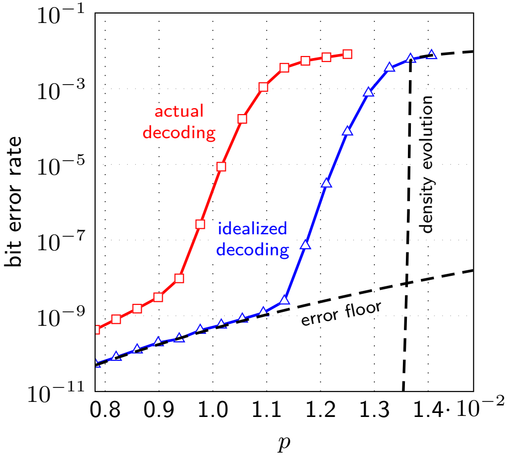
- Staircase code with $n = 256$ and $t = 2$



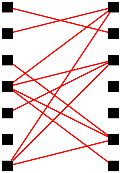
Performance Loss

- Staircase code with $n = 256$ and $t = 2$





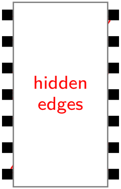
Anchor-Based Decoding



residual graph

0	0	0	0	0	1	1
1	1	0	1	1	0	1
0	1	0	0	0	0	1
1	1	1	0	1	1	0
0	0	1	0	0	0	1
1	0	1	1	1	1	0
0	1	0	1	0	1	1

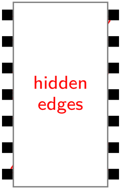
Anchor-Based Decoding



residual graph

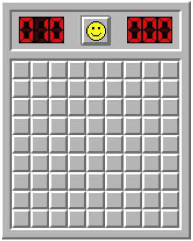
0	0	0	0	0	1	1
1	1	0	1	1	0	1
0	1	0	0	0	0	1
1	1	1	0	1	1	0
0	0	1	0	0	0	1
1	0	1	1	1	1	0
0	1	0	1	0	1	1

Anchor-Based Decoding

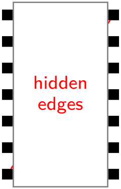


residual graph

0	0	0	0	0	1	1
1	1	0	1	1	0	1
0	1	0	0	0	0	1
1	1	1	0	1	1	0
0	0	1	0	0	0	1
1	0	1	1	1	1	0
0	1	0	1	0	1	1

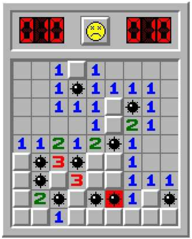


Anchor-Based Decoding

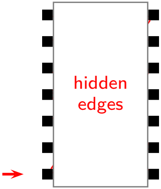


residual graph

0	0	0	0	0	1	1
1	1	0	1	1	0	1
0	1	0	0	0	0	1
1	1	1	0	1	1	0
0	0	1	0	0	0	1
1	0	1	1	1	1	0
0	1	0	1	0	1	1

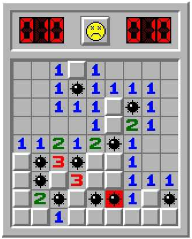


Anchor-Based Decoding

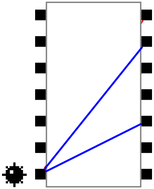


residual graph

0	0	0	0	0	1	1
1	1	0	1	1	0	1
0	1	0	0	0	0	1
1	1	1	0	1	1	0
0	0	1	0	0	0	1
1	0	1	1	1	1	0
0	1	0	1	0	1	1

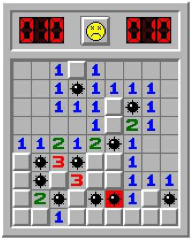


Anchor-Based Decoding

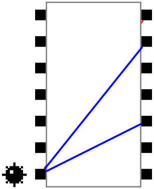


residual graph

0	0	0	0	0	1	1
1	1	0	1	1	0	0
0	1	0	0	0	0	1
1	1	1	0	1	1	0
0	0	1	0	0	0	0
1	0	1	1	1	1	0
0	1	0	1	0	1	1



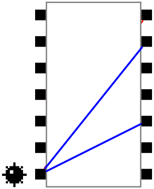
Anchor-Based Decoding



residual graph

0	0	0	0	0	1	1
1	1	0	1	1	0	0
0	1	0	0	0	0	1
1	1	1	0	1	1	0
0	0	1	0	0	0	0
1	0	1	1	1	1	0
0	1	0	1	0	1	1

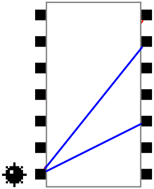
Anchor-Based Decoding


residual graph

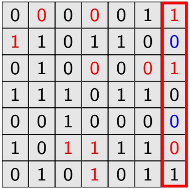
0	0	0	0	0	1	1
1	1	0	1	1	0	0
0	1	0	0	0	0	1
1	1	1	0	1	1	0
0	0	1	0	0	0	0
1	0	1	1	1	1	0
0	1	0	1	0	1	1

- Miscorrections lead to inconsistencies/conflicts: two component codewords may disagree on the value of a bit

Anchor-Based Decoding



residual graph



0	0	0	0	0	1	1
1	1	0	1	1	0	0
0	1	0	0	0	0	1
1	1	1	0	1	1	0
0	0	1	0	0	0	0
1	0	1	1	1	1	0
0	1	0	1	0	1	1

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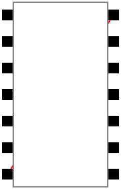
- Idea: make correctly decoded codewords anchors and trust their decisions (requires status information for each component codeword)

Anchor-Based Decoding



residual graph

0	0	0	0	0	1	1
1	1	0	1	1	0	1
0	1	0	0	0	0	1
1	1	1	0	1	1	0
0	0	1	0	0	0	1
1	0	1	1	1	1	0
0	1	0	1	0	1	1

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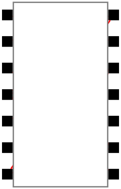
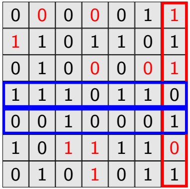


residual graph



0	0	0	0	0	1	1
1	1	0	1	1	0	1
0	1	0	0	0	0	1
1	1	1	0	1	1	0
0	0	1	0	0	0	1
1	0	1	1	1	1	0
0	1	0	1	0	1	1

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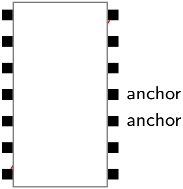

residual graph

0	0	0	0	0	1	1
1	1	0	1	1	0	1
0	1	0	0	0	0	1
1	1	1	0	1	1	0
0	0	1	0	0	0	1
1	0	1	1	1	1	0
0	1	0	1	0	1	1

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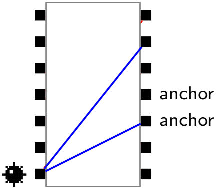
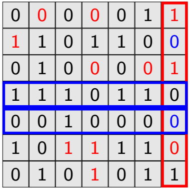


residual graph



0	0	0	0	0	1	1
1	1	0	1	1	0	0
0	1	0	0	0	0	1
1	1	1	0	1	1	0
0	0	1	0	0	0	0
1	0	1	1	1	1	0
0	1	0	1	0	1	1

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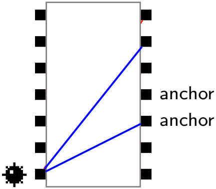
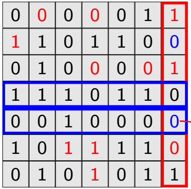
Anchor-Based Decoding



residual graph



0	0	0	0	0	1	1
1	1	0	1	1	0	0
0	1	0	0	0	0	1
1	1	1	0	1	1	0
0	0	1	0	0	0	0
1	0	1	1	1	1	0
0	1	0	1	0	1	1

conflict with anchor
$\implies$ reject bit flips

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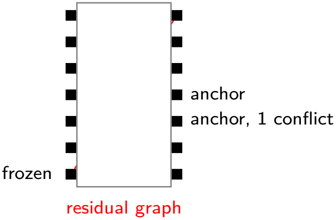

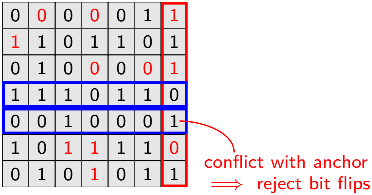

0	0	0	0	0	1	1
1	1	0	1	1	0	1
0	1	0	0	0	0	1
1	1	1	0	1	1	0
0	0	1	0	0	0	1
1	0	1	1	1	1	0
0	1	0	1	0	1	1

conflict with anchor
$\Longrightarrow$ reject bit flips

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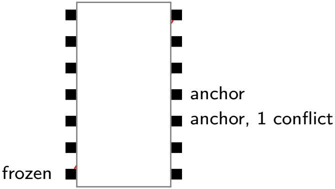
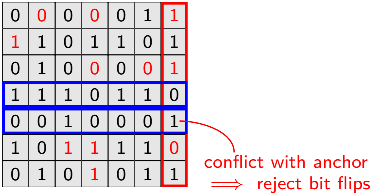
Anchor-Based Decoding



residual graph

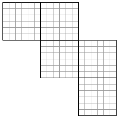


0	0	0	0	0	1	1
1	1	0	1	1	0	1
0	1	0	0	0	0	1
1	1	1	0	1	1	0
0	0	1	0	0	0	1
1	0	1	1	1	1	0
0	1	0	1	0	1	1

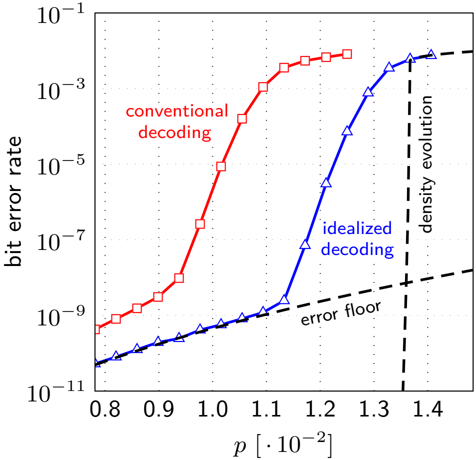
conflict with anchor
$\Longrightarrow$ reject bit flips

- Miscorrections lead to inconsistencies/conflicts: two component codewords may disagree on the value of a bit
- Idea: make correctly decoded codewords anchors and trust their decisions (requires status information for each component codeword)
- If any anchor has too many conflicts, backtrack its bit flips

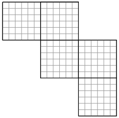
Simulation Results



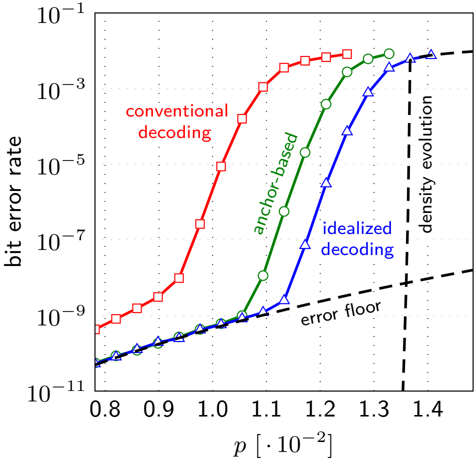
- Staircase code with $n = 256$ and $t = 2$



Simulation Results

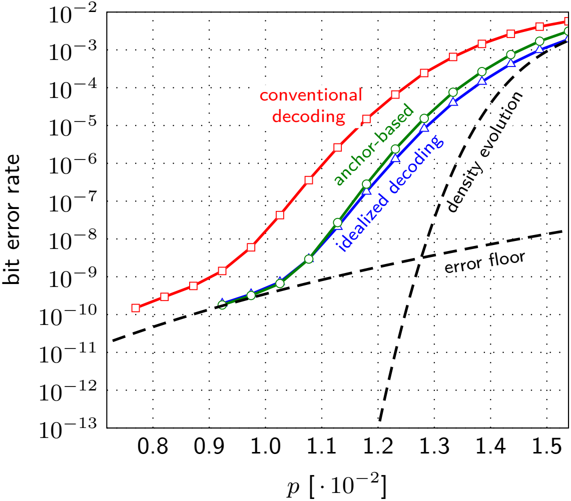


- Staircase code with $n = 256$ and $t = 2$



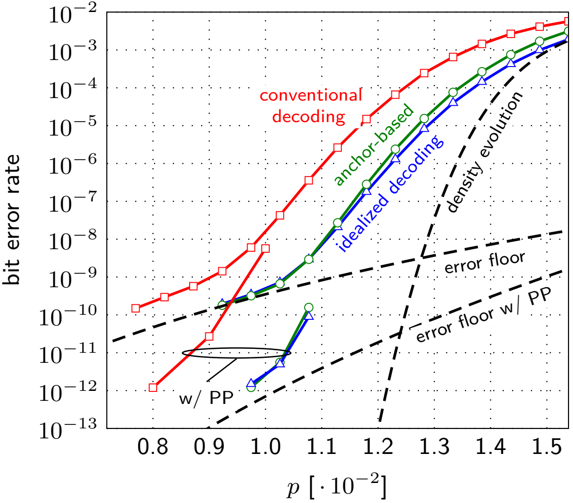
Simulation Results (cont.)

- Product code with $n = 195$ and $t = 2$, see [Condo et al., 2017]



Simulation Results (cont.)

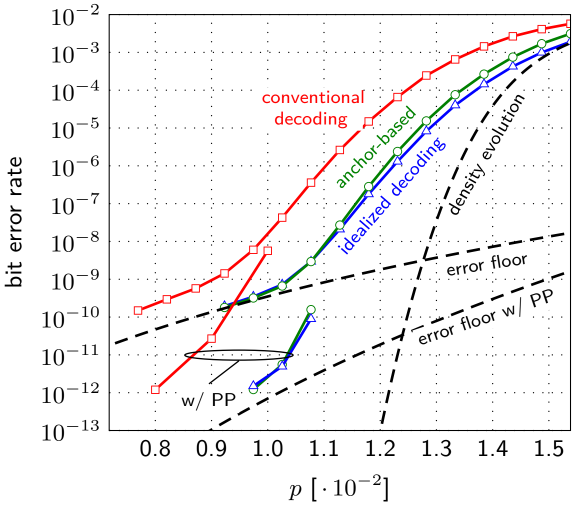
- Product code with $n = 195$ and $t = 2$, see [Condo et al., 2017]



post-processing (PP):
[Jian et al., 2014]
[Mittelholzer et al., 2016]
[Holzbaur et al., 2017]

Simulation Results (cont.)

- Product code with $n = 195$ and $t = 2$, see [Condo et al., 2017]



post-processing (PP):
[Jian et al., 2014]
[Mittelholzer et al., 2016]
[Holzbaur et al., 2017]

Future work: PP for staircase codes, complexity impact on product decoder architecture, ...

Part 1: Conclusions

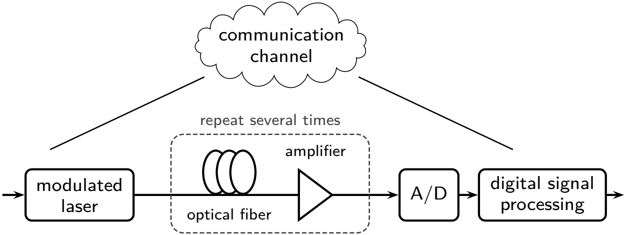
- Asymptotic density evolution analysis possible for many deterministic generalized product codes over the binary erasure channel
- In practice, miscorrection-free performance over the binary symmetric channel can be approached with anchor-based decoding

Part 2: Deep Learning

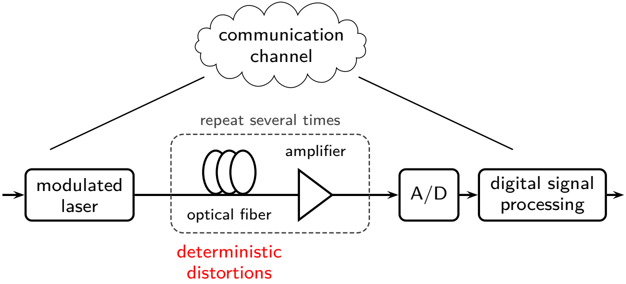
Deep Learning for Digital Backpropagation

communication channel

Deep Learning for Digital Backpropagation

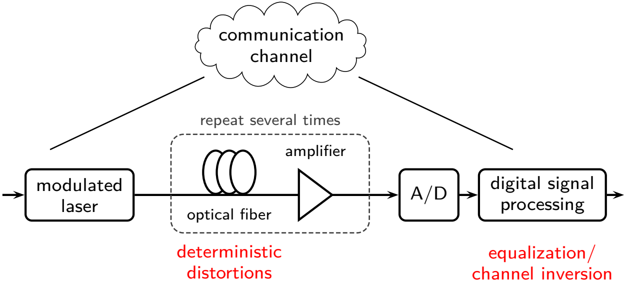


Deep Learning for Digital Backpropagation



- Dispersion: different wavelengths travel at different speeds (linear)
- Kerr effect: refractive index changes with signal intensity (nonlinear)

Deep Learning for Digital Backpropagation

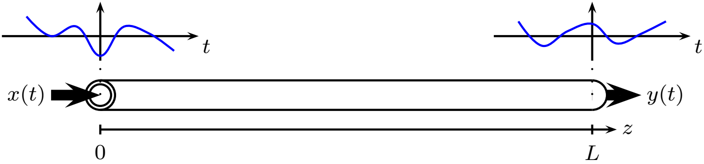

communication channel
repeat several times
amplifier
modulated laser
optical fiber
A/D
digital signal processing
deterministic distortions
equalization/ channel inversion

- Dispersion: different wavelengths travel at different speeds (linear)
- Kerr effect: refractive index changes with signal intensity (nonlinear)

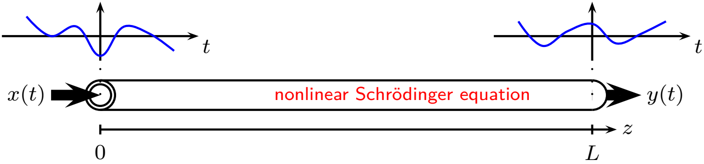
Outline: Part 2 (Deep Learning)

1. Channel modeling and digital backpropagation
2. Machine learning for complexity-reduced digital backpropagation

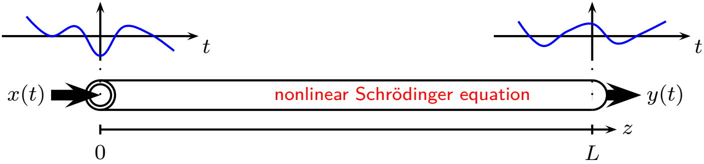
Deterministic Channel Modeling



Deterministic Channel Modeling

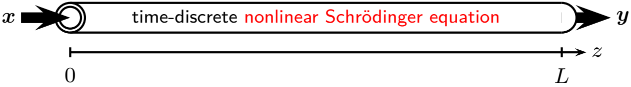


Deterministic Channel Modeling



- Sampling over a fixed time interval $\implies \mathcal{F} : \mathbb{C}^n \to \mathbb{C}^n$

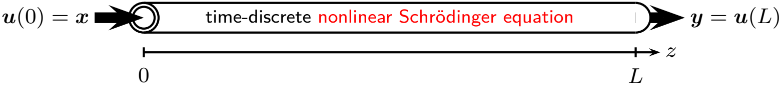
Deterministic Channel Modeling



- Sampling over a fixed time interval $\implies$ $\mathcal{F}: \mathbb{C}^n \to \mathbb{C}^n$

Deterministic Channel Modeling

$$\frac{\mathrm{d}\boldsymbol{u}(z)}{\mathrm{d}z} = \boldsymbol{A}\boldsymbol{u}(z) - \jmath\gamma\boldsymbol{\rho}(\boldsymbol{u}(z))$$


$\boldsymbol{u}(0) = \boldsymbol{x}$
time-discrete nonlinear Schrödinger equation
$\boldsymbol{y} = \boldsymbol{u}(L)$
z
0
L

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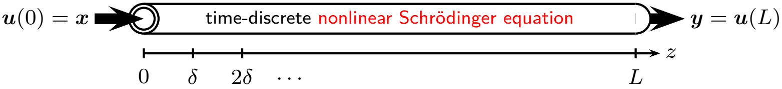
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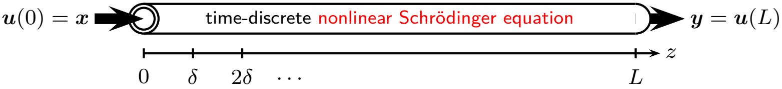

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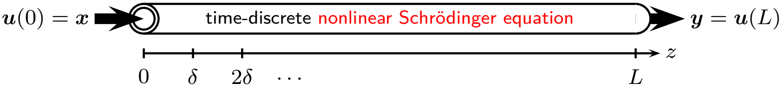
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L

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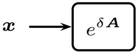

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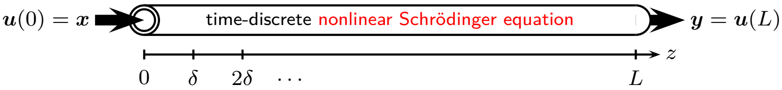
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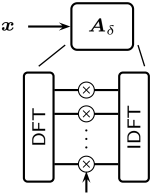
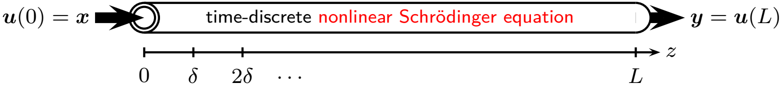


$H_k = e^{\jmath \frac{\beta_2}{2} \delta \omega_k^2}$ group velocity dispersion (all-pass filter)

Deterministic Channel Modeling

$$\frac{\mathrm{d}\boldsymbol{u}(z)}{\mathrm{d}z} = -\jmath\gamma\boldsymbol{\rho}(\boldsymbol{u}(z))$$

$\rho(x) = |x|^2 x$ element-wise



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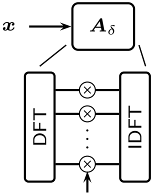

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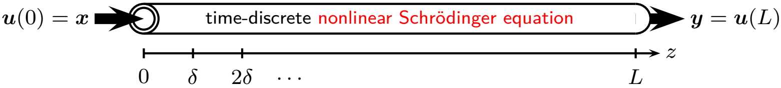

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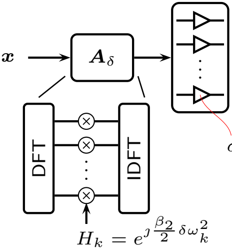

$\sigma_\delta(x) = xe^{-\jmath\gamma\delta|x|^2}$ Kerr effect

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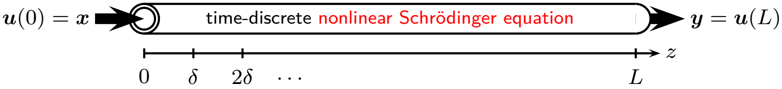
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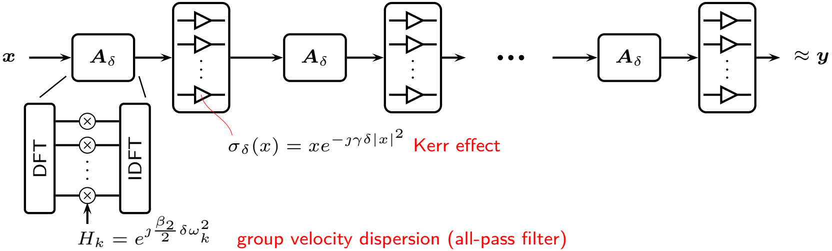
$\boldsymbol{x}$
$\boldsymbol{A}_\delta$
$\boldsymbol{A}_\delta$
$\cdots$
$\boldsymbol{A}_\delta$
$\approx \boldsymbol{y}$
DFT
IDFT
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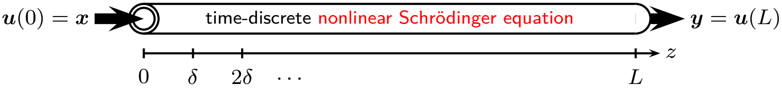

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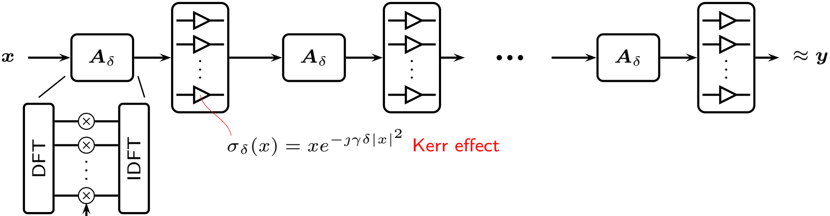

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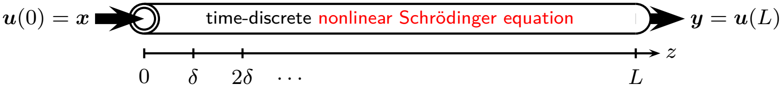
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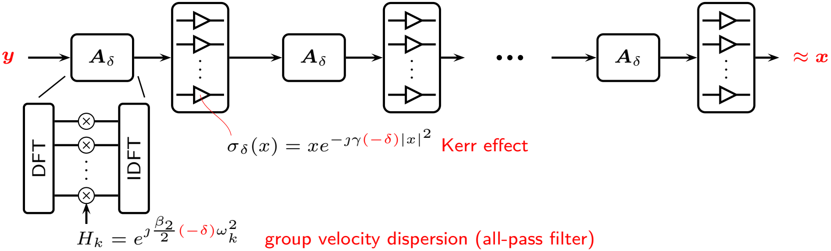

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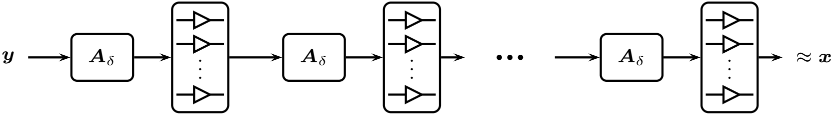
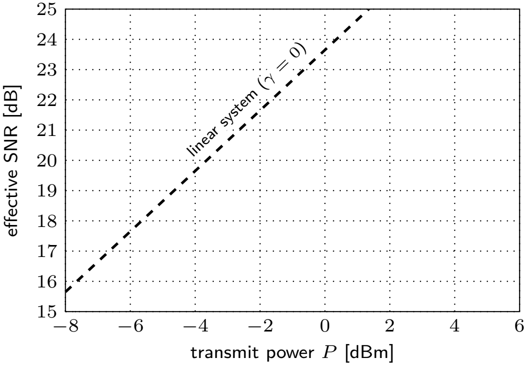
Performance of Digital Backpropagation





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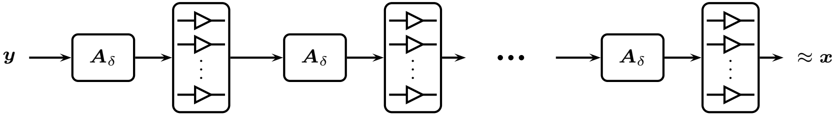

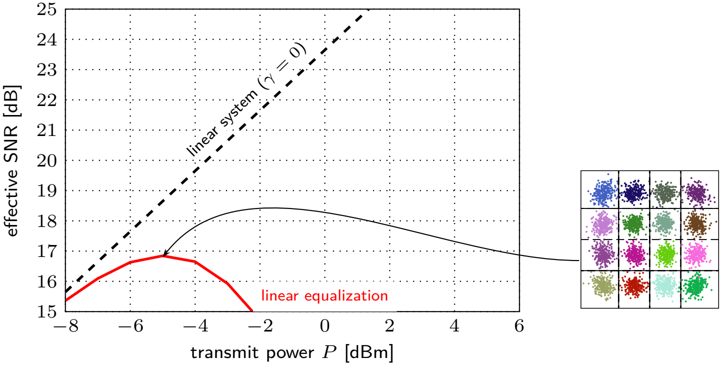

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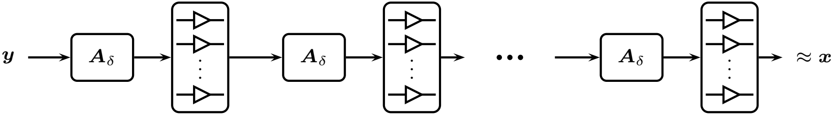
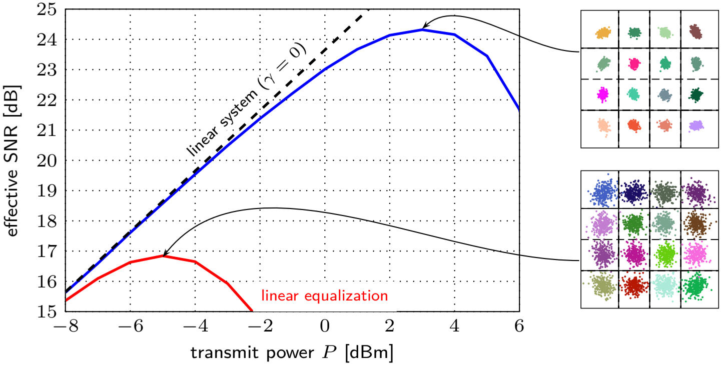
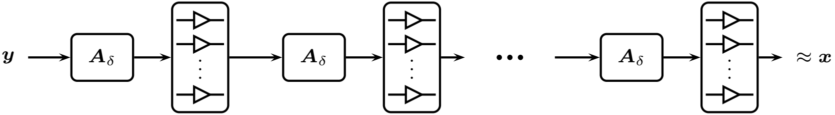




Complexity of the Split-Step Fourier Method



Split-step Fourier method:

```
for j = 1:M
  y = ifft(H.*fft(y)); % group velocity dispersion
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Linear equalization:

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y = ifft(Htilde.*fft(y)); % Htilde = H * H * ... * H
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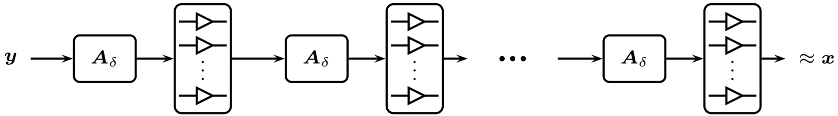

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At least M times more complex than linear equalization due to FFT/IFFT.

Example: $25 \times 80\,$km spans, 1 step per span $\implies$ $> 25\times$ increased complexity

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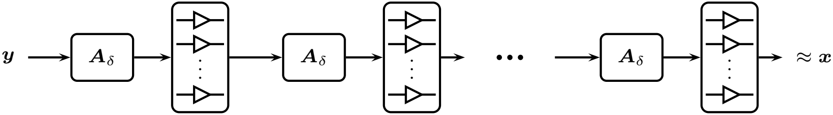

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Linear equalization: (already very power-hungry DSP block)

```
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At least M times more complex than linear equalization due to FFT/IFFT.

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Complexity-Reduced Digital Backpropagation

Literature (randomly sampled):

- "with only four steps for the entire link . . . " [Du and Lowery, 2010]
- "we report up to 80% reduction in required back-propagation steps" [Rafique et al., 2011]
- "one novel method is proposed to reduce the required stage number down to 1/4" [Li et al., 2011]
- "it reduces 85% back-propagation stages [. . .]" [Yan et al., 2011]
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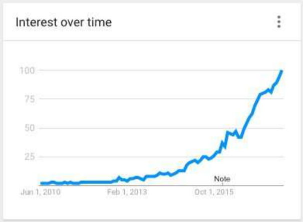
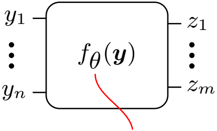


Google trends for "deep learning"

Are many steps really that inefficient?

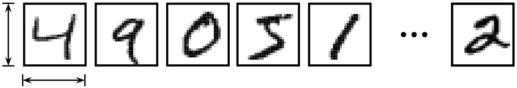
Supervised Learning



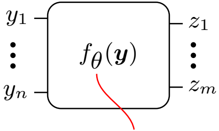
parameters
to be optimized/learned

Supervised Learning

handwritten digit recognition (MNIST: 70,000 images)



28×28 pixels $\implies n = 784$



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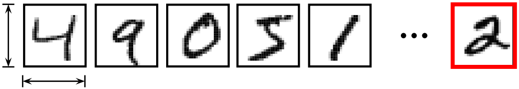

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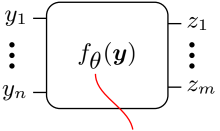

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$$\boldsymbol{z}\quad \begin{array}{c} 0.01 \\ 0.92 \\ 0.01 \\ 0.00 \\ 0.00 \\ 0.01 \\ 0.00 \\ 0.04 \\ 0.01 \\ 0.01 \end{array}$$

Supervised Learning

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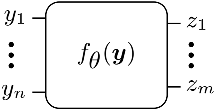

$\boldsymbol{z}$

0.01
0.92
0.01
0.00
0.00
0.01
0.00
0.04
0.01
0.01

How to choose $f_\theta(\boldsymbol{y})$? Deep feed-forward neural networks

Supervised Learning

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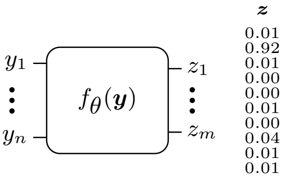

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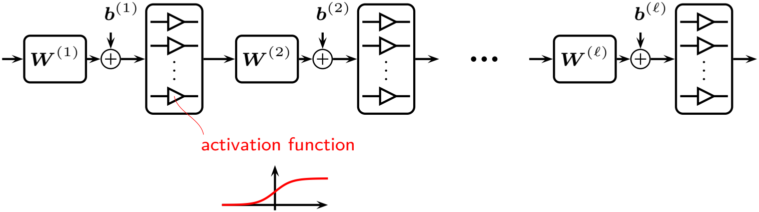

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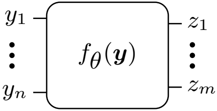
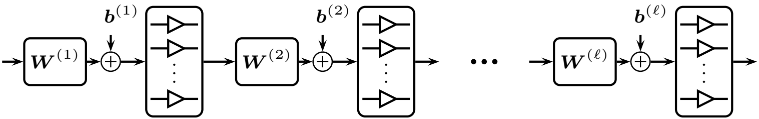


$\boldsymbol{z}$	$\boldsymbol{x}$
0.01	0
0.92	1
0.01	0
0.00	0
0.00	0
0.01	0
0.00	0
0.04	0
0.01	0
0.01	0

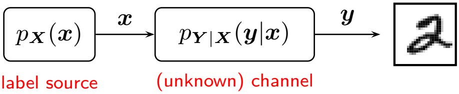
How to choose $f_\theta(\boldsymbol{y})$? Deep feed-forward neural networks

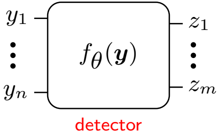


How to optimize $\theta = \{\boldsymbol{W}^{(1)}, \dots, \boldsymbol{W}^{(\ell)}, \boldsymbol{b}^{(1)}, \dots, \boldsymbol{b}^{(\ell)}\}$? Deep learning

$$\min_{\theta} \sum_{i=1}^{N} \text{Loss}(f_\theta(\boldsymbol{y}^{(i)}), \boldsymbol{x}^{(i)}) \triangleq g(\theta) \qquad \text{using} \quad \theta_{k+1} = \theta_k - \lambda \nabla_\theta g(\theta_k) \qquad (1)$$

Supervised Learning


$p_{\boldsymbol{X}}(\boldsymbol{x})$
$\boldsymbol{x}$
$p_{\boldsymbol{Y}|\boldsymbol{X}}(\boldsymbol{y}|\boldsymbol{x})$
$\boldsymbol{y}$
label source
(unknown) channel



$\boldsymbol{z}$	$\boldsymbol{x}$
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0.01	0
0.00	0
0.00	0
0.01	0
0.00	0
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0.01	0
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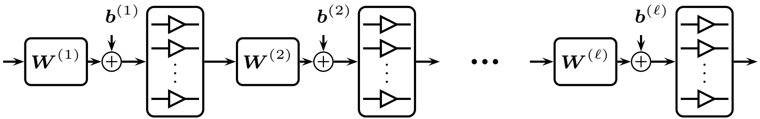

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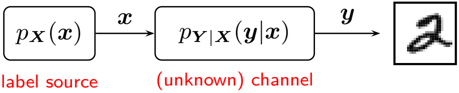
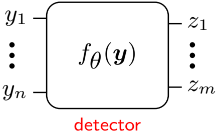
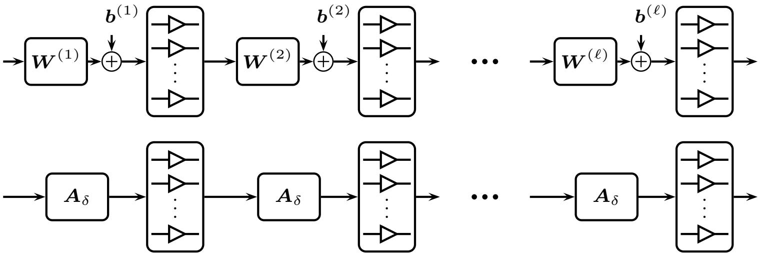
Supervised Learning


$p_{\boldsymbol{X}}(\boldsymbol{x})$
$\boldsymbol{x}$
$p_{\boldsymbol{Y}|\boldsymbol{X}}(\boldsymbol{y}|\boldsymbol{x})$
$\boldsymbol{y}$
label source
(unknown) channel

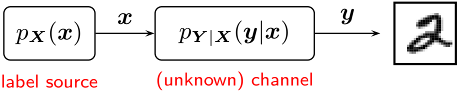


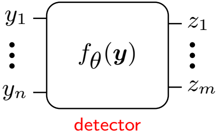
$\boldsymbol{z}$	$\boldsymbol{x}$
0.01	0
0.92	1
0.01	0
0.00	0
0.00	0
0.01	0
0.00	0
0.04	0
0.01	0
0.01	0

How to choose $f_\theta(\boldsymbol{y})$? Deep feed-forward neural networks



Supervised Learning


$p_{\boldsymbol{X}}(\boldsymbol{x})$
$\boldsymbol{x}$
$p_{\boldsymbol{Y}|\boldsymbol{X}}(\boldsymbol{y}|\boldsymbol{x})$
$\boldsymbol{y}$
label source
(unknown) channel



$\boldsymbol{z}$	$\boldsymbol{x}$
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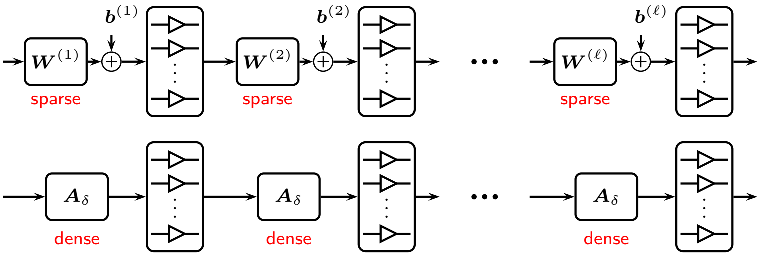
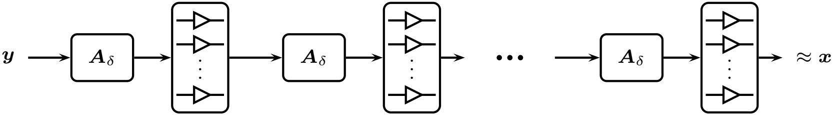
0.01	0
0.01	0

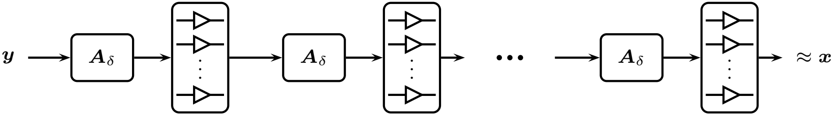
How to choose $f_\theta(\boldsymbol{y})$? Deep feed-forward neural networks

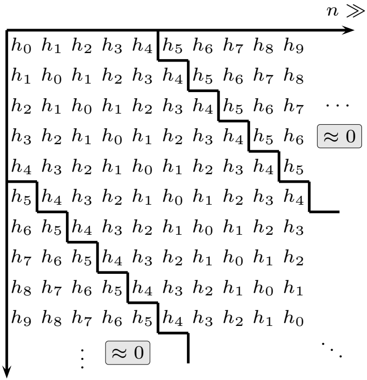


Truncation


$\boldsymbol{y}$
$\boldsymbol{A}_\delta$
$\boldsymbol{A}_\delta$
$\boldsymbol{A}_\delta$
$\approx \boldsymbol{x}$

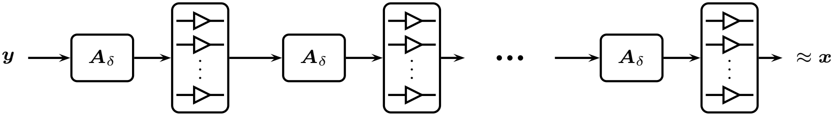
Truncation





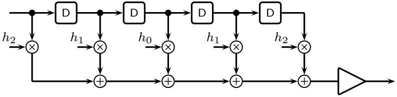
$$\begin{array}{cccccccccc}
h_0 & h_1 & h_2 & h_3 & h_4 & h_5 & h_6 & h_7 & h_8 & h_9 \\
h_1 & h_0 & h_1 & h_2 & h_3 & h_4 & h_5 & h_6 & h_7 & h_8 \\
h_2 & h_1 & h_0 & h_1 & h_2 & h_3 & h_4 & h_5 & h_6 & h_7 \\
h_3 & h_2 & h_1 & h_0 & h_1 & h_2 & h_3 & h_4 & h_5 & h_6 \\
h_4 & h_3 & h_2 & h_1 & h_0 & h_1 & h_2 & h_3 & h_4 & h_5 \\
h_5 & h_4 & h_3 & h_2 & h_1 & h_0 & h_1 & h_2 & h_3 & h_4 \\
h_6 & h_5 & h_4 & h_3 & h_2 & h_1 & h_0 & h_1 & h_2 & h_3 \\
h_7 & h_6 & h_5 & h_4 & h_3 & h_2 & h_1 & h_0 & h_1 & h_2 \\
h_8 & h_7 & h_6 & h_5 & h_4 & h_3 & h_2 & h_1 & h_0 & h_1 \\
h_9 & h_8 & h_7 & h_6 & h_5 & h_4 & h_3 & h_2 & h_1 & h_0
\end{array}$$

Truncation

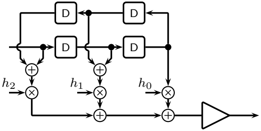


$n \gg 9$

$$\begin{array}{cccccccccc}
h_0 & h_1 & h_2 & h_3 & h_4 & 0 & 0 & 0 & 0 & 0 \\
h_1 & h_0 & h_1 & h_2 & h_3 & h_4 & 0 & 0 & 0 & 0 \\
h_2 & h_1 & h_0 & h_1 & h_2 & h_3 & h_4 & 0 & 0 & 0 & \cdots \\
h_3 & h_2 & h_1 & h_0 & h_1 & h_2 & h_3 & h_4 & 0 & 0 \\
h_4 & h_3 & h_2 & h_1 & h_0 & h_1 & h_2 & h_3 & h_4 & 0 \\
0 & h_4 & h_3 & h_2 & h_1 & h_0 & h_1 & h_2 & h_3 & h_4 \\
0 & 0 & h_4 & h_3 & h_2 & h_1 & h_0 & h_1 & h_2 & h_3 \\
0 & 0 & 0 & h_4 & h_3 & h_2 & h_1 & h_0 & h_1 & h_2 \\
0 & 0 & 0 & 0 & h_4 & h_3 & h_2 & h_1 & h_0 & h_1 \\
0 & 0 & 0 & 0 & 0 & h_4 & h_3 & h_2 & h_1 & h_0 \\
 & & \vdots & & & & & & & & \ddots
\end{array}$$



finite impulse response (FIR) filter



symmetric filter coefficients
$\Longrightarrow$ folded implementation

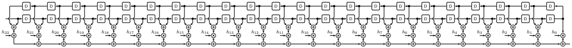
Time-Domain Digital Backpropagation

Complexity estimate in [Ip and Kahn, 2008] for $25 \times 80\,\text{km}$ using filters

Time-Domain Digital Backpropagation

Complexity estimate in [Ip and Kahn, 2008] for 25×80 km using filters

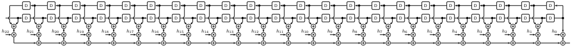
Linear equalization (47 taps for 2000 km):



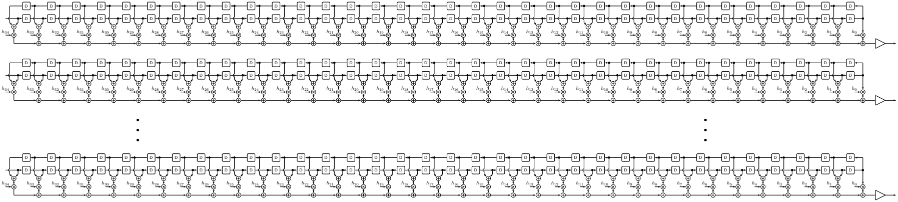
Time-Domain Digital Backpropagation

Complexity estimate in [Ip and Kahn, 2008] for $25 \times 80\,\mathrm{km}$ using filters

Linear equalization (47 taps for $2000\,\mathrm{km}$):

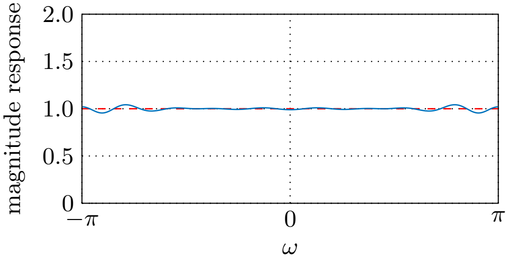


Digital backpropagation (25 times 70 taps for $80\,\mathrm{km}$):



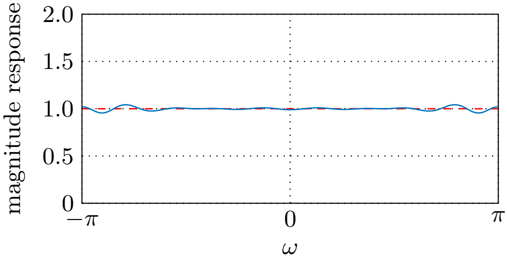
$\Longrightarrow$ > 100 times more operations per data symbol

Truncation Errors

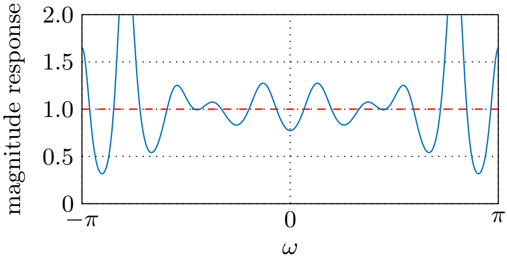


$$\boldsymbol{h}^{(1)} = \boldsymbol{h}^{(2)} = \cdots = \boldsymbol{h}^{(25)}$$

Truncation Errors

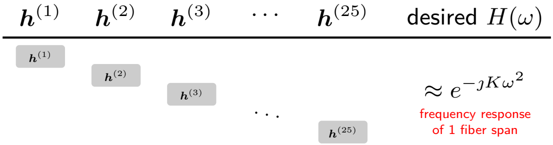


$$\boldsymbol{h}^{(1)} = \boldsymbol{h}^{(2)} = \cdots = \boldsymbol{h}^{(25)}$$

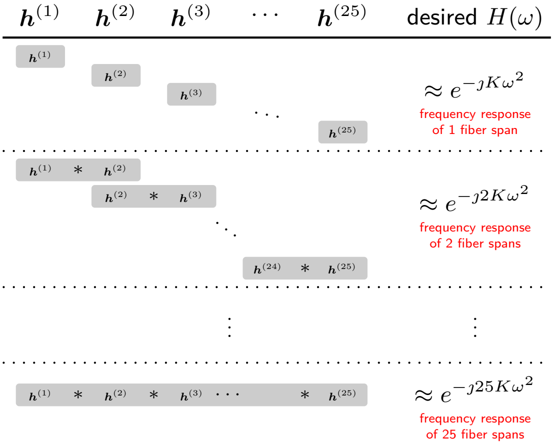


$$\boldsymbol{h}^{(1)} * \boldsymbol{h}^{(2)} * \cdots * \boldsymbol{h}^{(25)}$$

Joint Filter Optimization Problem



Joint Filter Optimization Problem



$\boldsymbol{h}^{(1)}$ $\boldsymbol{h}^{(2)}$ $\boldsymbol{h}^{(3)}$ $\cdots$ $\boldsymbol{h}^{(25)}$ desired $H(\omega)$

$\boldsymbol{h}^{(1)}$, $\boldsymbol{h}^{(2)}$, $\boldsymbol{h}^{(3)}$, $\ddots$, $\boldsymbol{h}^{(25)}$

$\approx e^{-\jmath K\omega^2}$

frequency response of 1 fiber span

$\boldsymbol{h}^{(1)} * \boldsymbol{h}^{(2)}$, $\boldsymbol{h}^{(2)} * \boldsymbol{h}^{(3)}$, $\ddots$, $\boldsymbol{h}^{(24)} * \boldsymbol{h}^{(25)}$

$\approx e^{-\jmath 2K\omega^2}$

frequency response of 2 fiber spans

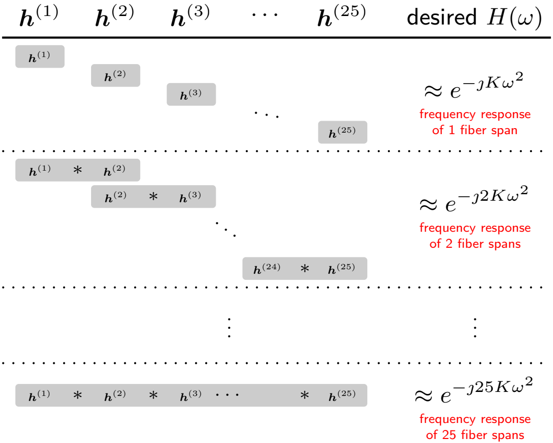
$\vdots$ $\vdots$

$\boldsymbol{h}^{(1)} * \boldsymbol{h}^{(2)} * \boldsymbol{h}^{(3)} \cdots * \boldsymbol{h}^{(25)}$

$\approx e^{-\jmath 25K\omega^2}$

frequency response of 25 fiber spans

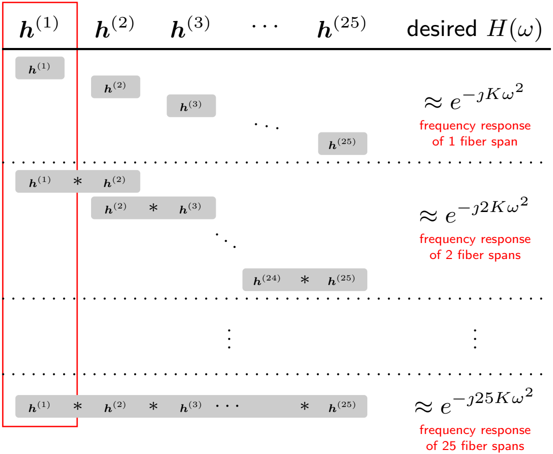
Joint Filter Optimization Problem


$\boldsymbol{h}^{(1)}$ $\boldsymbol{h}^{(2)}$ $\boldsymbol{h}^{(3)}$ $\cdots$ $\boldsymbol{h}^{(25)}$ desired $H(\omega)$
$\boldsymbol{h}^{(1)}$
$\boldsymbol{h}^{(2)}$
$\boldsymbol{h}^{(3)}$
$\ddots$
$\boldsymbol{h}^{(25)}$
$\approx e^{-\jmath K\omega^2}$
frequency response of 1 fiber span
$\boldsymbol{h}^{(1)} * \boldsymbol{h}^{(2)}$
$\boldsymbol{h}^{(2)} * \boldsymbol{h}^{(3)}$
$\ddots$
$\boldsymbol{h}^{(24)} * \boldsymbol{h}^{(25)}$
$\approx e^{-\jmath 2K\omega^2}$
frequency response of 2 fiber spans
$\vdots$
$\vdots$
$\boldsymbol{h}^{(1)} * \boldsymbol{h}^{(2)} * \boldsymbol{h}^{(3)} \cdots * \boldsymbol{h}^{(25)}$
$\approx e^{-\jmath 25K\omega^2}$
frequency response of 25 fiber spans

Our approach

1. Iterative least-squares
2. Use solution as θ_0 for deep learning

Joint Filter Optimization Problem



$\boldsymbol{h}^{(1)}$ $\boldsymbol{h}^{(2)}$ $\boldsymbol{h}^{(3)}$ $\cdots$ $\boldsymbol{h}^{(25)}$ desired $H(\omega)$

$\boldsymbol{h}^{(1)}$ $\boldsymbol{h}^{(2)}$ $\boldsymbol{h}^{(3)}$ $\ddots$ $\boldsymbol{h}^{(25)}$

$\approx e^{-\jmath K\omega^2}$

frequency response of 1 fiber span

$\boldsymbol{h}^{(1)} * \boldsymbol{h}^{(2)}$

$\boldsymbol{h}^{(2)} * \boldsymbol{h}^{(3)}$

$\ddots$

$\boldsymbol{h}^{(24)} * \boldsymbol{h}^{(25)}$

$\approx e^{-\jmath 2K\omega^2}$

frequency response of 2 fiber spans

$\vdots$ $\vdots$

$\boldsymbol{h}^{(1)} * \boldsymbol{h}^{(2)} * \boldsymbol{h}^{(3)} \cdots * \boldsymbol{h}^{(25)}$

$\approx e^{-\jmath 25K\omega^2}$

frequency response of 25 fiber spans

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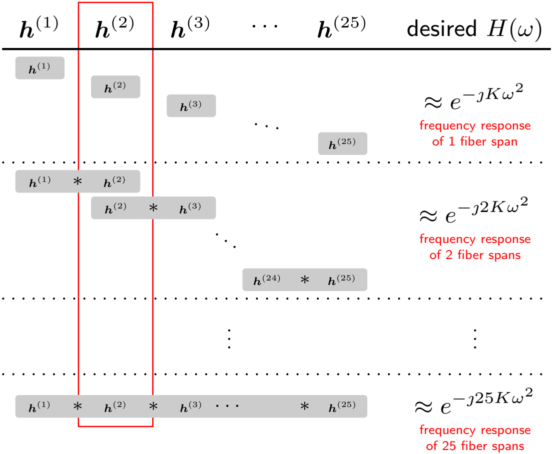

$\boldsymbol{h}^{(1)}$ $\boldsymbol{h}^{(2)}$ $\boldsymbol{h}^{(3)}$ $\cdots$ $\boldsymbol{h}^{(25)}$ desired $H(\omega)$

$\boldsymbol{h}^{(1)}$ $\boldsymbol{h}^{(2)}$ $\boldsymbol{h}^{(3)}$ $\ddots$ $\boldsymbol{h}^{(25)}$ $\approx e^{-\jmath K\omega^2}$

frequency response
of 1 fiber span

$\boldsymbol{h}^{(1)} * \boldsymbol{h}^{(2)}$ $\boldsymbol{h}^{(2)} * \boldsymbol{h}^{(3)}$ $\ddots$ $\boldsymbol{h}^{(24)} * \boldsymbol{h}^{(25)}$ $\approx e^{-\jmath 2K\omega^2}$

frequency response
of 2 fiber spans

$\vdots$ $\vdots$

$\boldsymbol{h}^{(1)} * \boldsymbol{h}^{(2)} * \boldsymbol{h}^{(3)} \cdots * \boldsymbol{h}^{(25)}$ $\approx e^{-\jmath 25K\omega^2}$

frequency response
of 25 fiber spans

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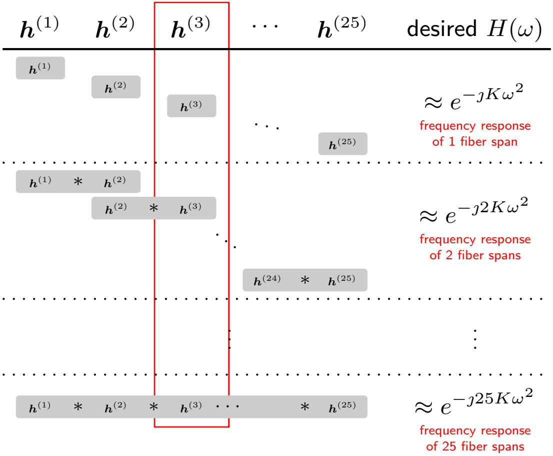
Joint Filter Optimization Problem



$\boldsymbol{h}^{(1)}$ $\boldsymbol{h}^{(2)}$ $\boldsymbol{h}^{(3)}$ $\cdots$ $\boldsymbol{h}^{(25)}$ desired $H(\omega)$

$\boldsymbol{h}^{(1)}$, $\boldsymbol{h}^{(2)}$, $\boldsymbol{h}^{(3)}$, $\ddots$, $\boldsymbol{h}^{(25)}$ $\approx e^{-\jmath K \omega^2}$ frequency response of 1 fiber span

$\boldsymbol{h}^{(1)} * \boldsymbol{h}^{(2)}$, $\boldsymbol{h}^{(2)} * \boldsymbol{h}^{(3)}$, $\ddots$, $\boldsymbol{h}^{(24)} * \boldsymbol{h}^{(25)}$ $\approx e^{-\jmath 2K \omega^2}$ frequency response of 2 fiber spans

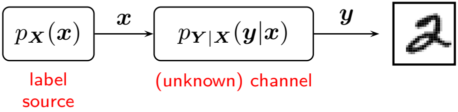
$\vdots$ $\vdots$

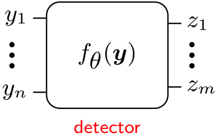
$\boldsymbol{h}^{(1)} * \boldsymbol{h}^{(2)} * \boldsymbol{h}^{(3)} \cdots * \boldsymbol{h}^{(25)}$ $\approx e^{-\jmath 25K \omega^2}$ frequency response of 25 fiber spans

Our approach

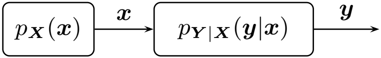
1. Iterative least-squares
2. Use solution as θ_0 for deep learning

Learned Digital Backpropagation


$p_{\boldsymbol{X}}(\boldsymbol{x})$
$\boldsymbol{x}$
$p_{\boldsymbol{Y}|\boldsymbol{X}}(\boldsymbol{y}|\boldsymbol{x})$
$\boldsymbol{y}$
label source
(unknown) channel

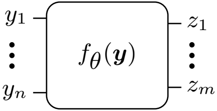


Learned Digital Backpropagation


$p_{\boldsymbol{X}}(\boldsymbol{x})$
$\boldsymbol{x}$
$p_{\boldsymbol{Y}|\boldsymbol{X}}(\boldsymbol{y}|\boldsymbol{x})$
$\boldsymbol{y}$

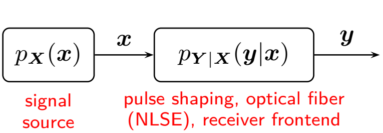
signal source

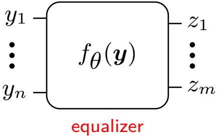
pulse shaping, optical fiber (NLSE), receiver frontend



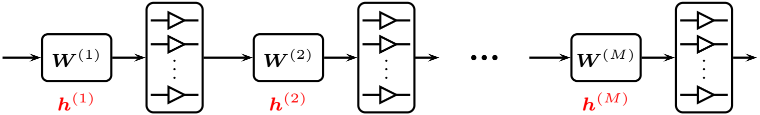
equalizer

Learned Digital Backpropagation

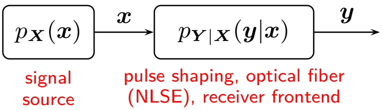

$p_{\boldsymbol{X}}(\boldsymbol{x})$
$\boldsymbol{x}$
$p_{\boldsymbol{Y}|\boldsymbol{X}}(\boldsymbol{y}|\boldsymbol{x})$
$\boldsymbol{y}$
signal source
pulse shaping, optical fiber (NLSE), receiver frontend



$f_\theta(\boldsymbol{y})$: TensorFlow implementation of the computation graph



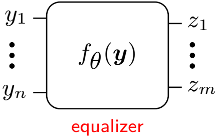
Learned Digital Backpropagation



$p_{\boldsymbol{X}}(\boldsymbol{x})$ $\xrightarrow{\boldsymbol{x}}$ $p_{\boldsymbol{Y}|\boldsymbol{X}}(\boldsymbol{y}|\boldsymbol{x})$ $\xrightarrow{\boldsymbol{y}}$

signal source

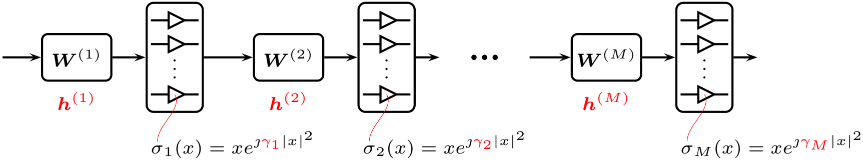
pulse shaping, optical fiber (NLSE), receiver frontend



y_1 … y_n $f_\theta(\boldsymbol{y})$ z_1 … z_m

equalizer

$f_\theta(\boldsymbol{y})$: TensorFlow implementation of the computation graph



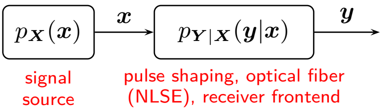
$\boldsymbol{W}^{(1)}$ $\boldsymbol{h}^{(1)}$ $\boldsymbol{W}^{(2)}$ $\boldsymbol{h}^{(2)}$ $\cdots$ $\boldsymbol{W}^{(M)}$ $\boldsymbol{h}^{(M)}$

$\sigma_1(x) = x e^{\jmath \gamma_1 |x|^2}$

$\sigma_2(x) = x e^{\jmath \gamma_2 |x|^2}$

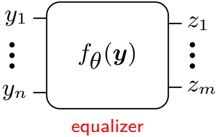
$\sigma_M(x) = x e^{\jmath \gamma_M |x|^2}$

Learned Digital Backpropagation


$p_{\boldsymbol{X}}(\boldsymbol{x})$
$\boldsymbol{x}$
$p_{\boldsymbol{Y}|\boldsymbol{X}}(\boldsymbol{y}|\boldsymbol{x})$
$\boldsymbol{y}$

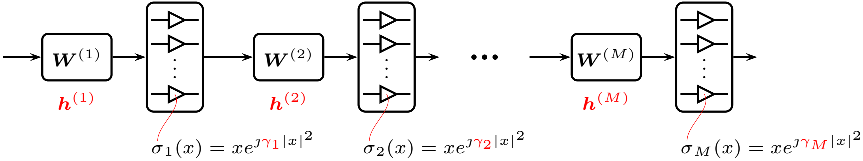
signal source

pulse shaping, optical fiber (NLSE), receiver frontend



equalizer

$f_\theta(\boldsymbol{y})$: TensorFlow implementation of the computation graph



$\sigma_1(x) = xe^{\jmath\gamma_1|x|^2}$ $\sigma_2(x) = xe^{\jmath\gamma_2|x|^2}$ $\sigma_M(x) = xe^{\jmath\gamma_M|x|^2}$

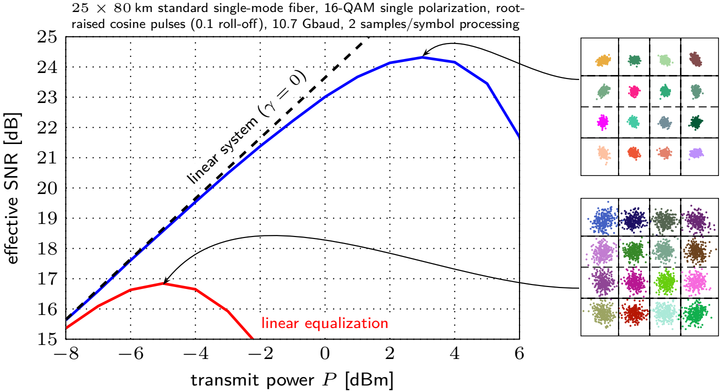
Deep learning of parameters $\theta = \{\boldsymbol{W}^{(1)}, \dots, \boldsymbol{W}^{(M)}, \gamma_1, \dots, \gamma_M\}$

$$\min_\theta \sum_{i=1}^{N} \text{Loss}(f_\theta(\boldsymbol{y}^{(i)}), \boldsymbol{x}^{(i)}) \triangleq g(\theta) \qquad \text{using} \quad \theta_{k+1} = \theta_k - \lambda \nabla_\theta g(\theta_k)$$

mean squared error

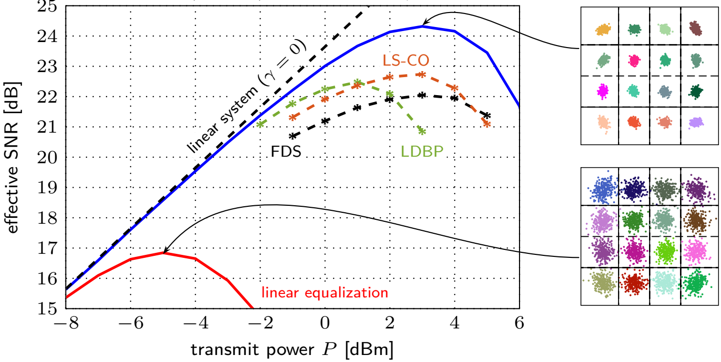
Adam optimizer, learning rate $\lambda = 0.001$

Results


25×80 km standard single-mode fiber, 16-QAM single polarization, root-raised cosine pulses (0.1 roll-off), 10.7 Gbaud, 2 samples/symbol processing
effective SNR [dB]
25
24
23
22
21
20
19
18
17
16
15
−8
−6
−4
−2
0
2
4
6
transmit power P [dBm]
linear system ($\gamma = 0$)
linear equalization

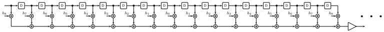
Results

25×80 km standard single-mode fiber, 16-QAM single polarization, root-raised cosine pulses (0.1 roll-off), 10.7 Gbaud, 2 samples/symbol processing



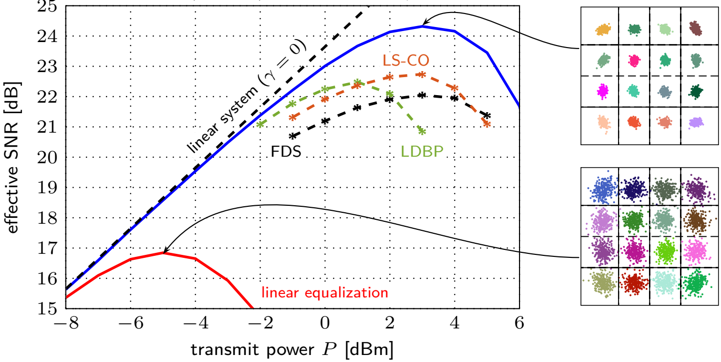
Results

FDS



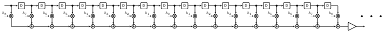
frequency-domain sampling (17 taps)

25 × 80 km standard single-mode fiber, 16-QAM single polarization, root-raised cosine pulses (0.1 roll-off), 10.7 Gbaud, 2 samples/symbol processing



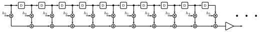
Results

FDS



frequency-domain sampling (17 taps)

LS-CO



least-squares [Eghbali et al., 2014], constrained out-of-band gain [Fougstedt et al., 2015], [Sheikh et al., 2016] (11 taps)

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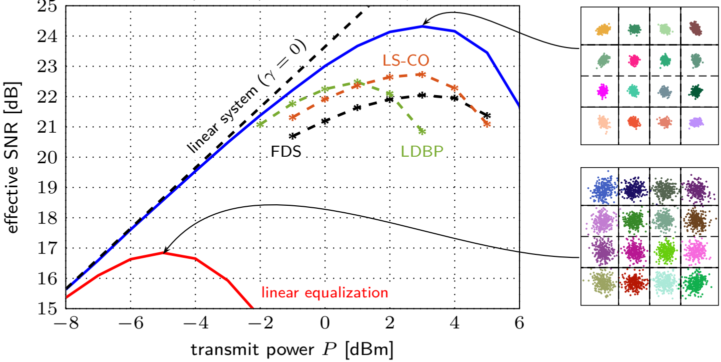

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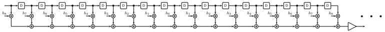

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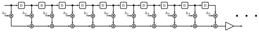

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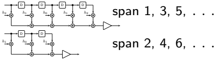
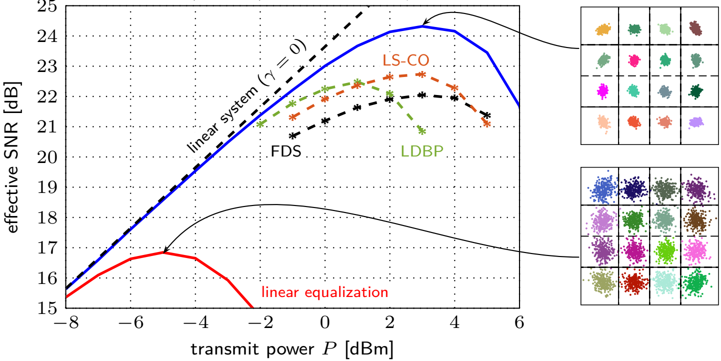
LDBP



learned digital backpropagation (alternate 5 and 3 taps)

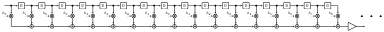
25 × 80 km standard single-mode fiber, 16-QAM single polarization, root-raised cosine pulses (0.1 roll-off), 10.7 Gbaud, 2 samples/symbol processing



Results

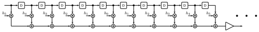
multiplications for the
linear step per span

FDS 8



. . . frequency-domain sampling (17 taps)

LS-CO 5



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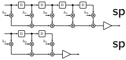
LDBP ≈ 1.5

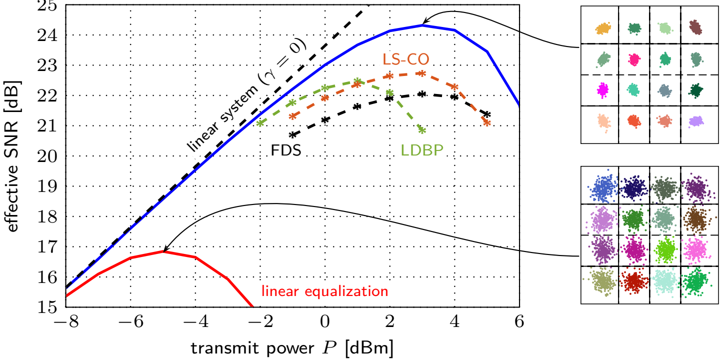


span 1, 3, 5, . . .

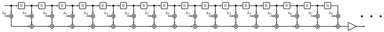
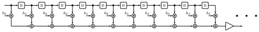
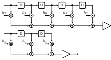
span 2, 4, 6, . . .

learned digital backpropagation (alternate 5 and 3 taps)
≈ 2× more multiplications than linear equalization

25 × 80 km standard single-mode fiber, 16-QAM single polarization, root-raised cosine pulses (0.1 roll-off), 10.7 Gbaud, 2 samples/symbol processing



Results

	multiplications for the linear step per span		
FDS	8		frequency-domain sampling (17 taps)
LS-CO	5		least-squares [Eghbali et al., 2014], constrained out-of-band gain [Fougstedt et al., 2015], [Sheikh et al., 2016] (11 taps)
LDBP	≈ 1.5	 span 1, 3, 5, . . . span 2, 4, 6, . . .	learned digital backpropagation (alternate 5 and 3 taps) ≈ 2× more multiplications than linear equalization

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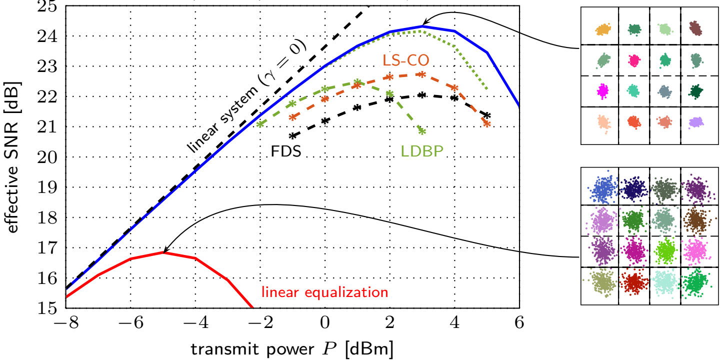
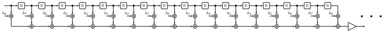
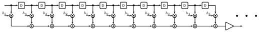
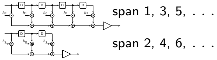


Results

	multiplications for the linear step per span		
FDS	8		frequency-domain sampling (17 taps)
LS-CO	5		least-squares [Eghbali et al., 2014], constrained out-of-band gain [Fougstedt et al., 2015], [Sheikh et al., 2016] (11 taps)
LDBP	≈ 1.5	 	learned digital backpropagation (alternate 5 and 3 taps) ≈ 2× more multiplications than linear equalization

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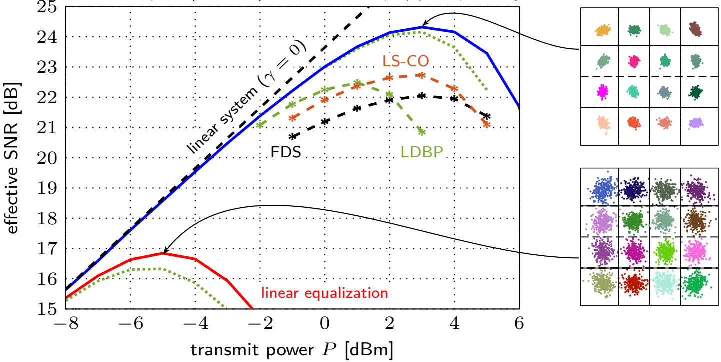

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Thank you!

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