A Deterministic Construction and Density Evolution Analysis for Generalized Product Codes

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In This Talk ...

- Deterministic code construction that recovers product codes, staircase codes, and block-wise braided codes as special cases
- Rigorous density evolution analysis possible over the binary erasure channel
- Application: Spatially-coupled product codes and symmetric generalized product codes

Introduction: Product Codes and Staircase Codes

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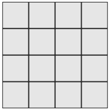
rectangular array [Elias, 1954]



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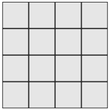

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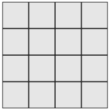

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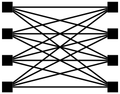

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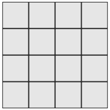

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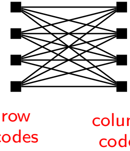

constraint node (CN) degree = component code length

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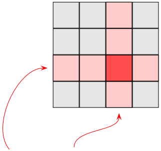

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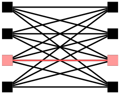

row codes

column codes

constraint node (CN) degree = component code length

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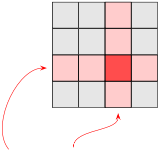

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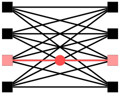

edge = degree-2 variable node (VN)

constraint node (CN) degree = component code length

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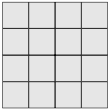

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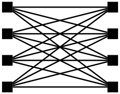

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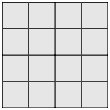

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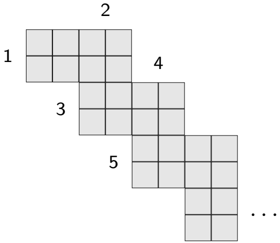

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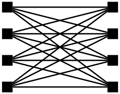

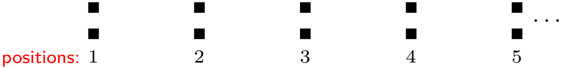

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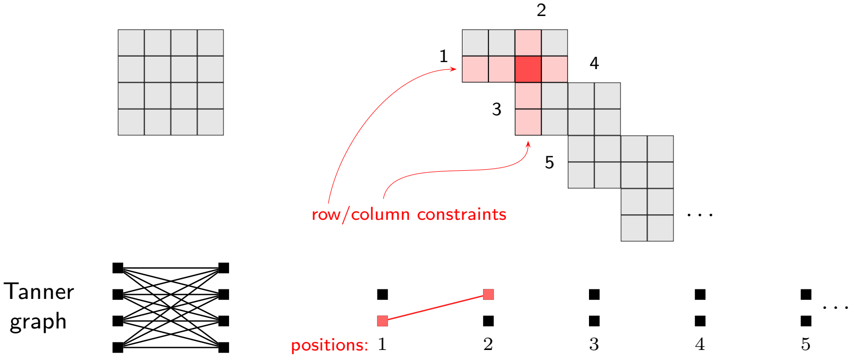

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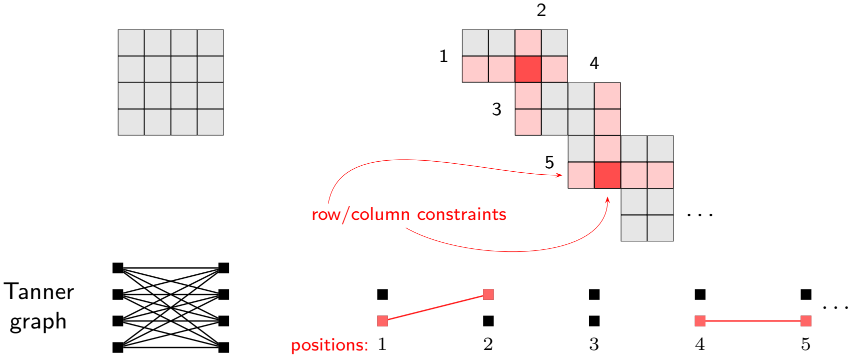

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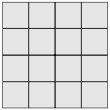

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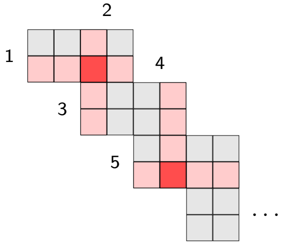

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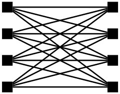

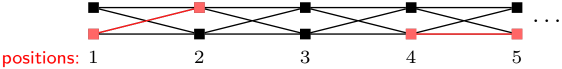

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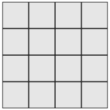

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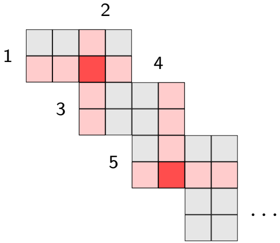

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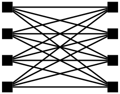

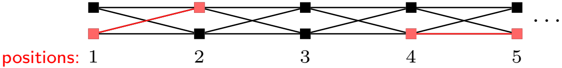

- Deterministic codes with fixed and structured Tanner graph

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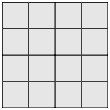

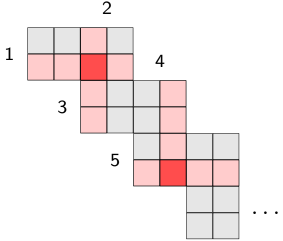

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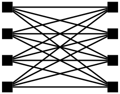

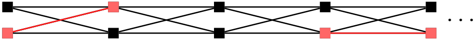

- Deterministic codes with fixed and structured Tanner graph
- Our code construction recovers these (and other) codes as special cases

Deterministic Construction for Generalized Product Codes

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- $\boldsymbol{\eta}$: binary symmetric $L \times L$ matrix (defines Tanner graph connectivity)

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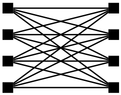

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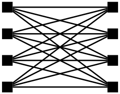

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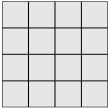

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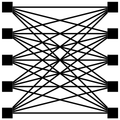
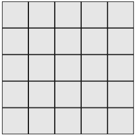
positions: 1 2 $n = 5 \implies d = 5$





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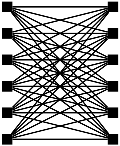
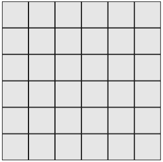
positions: 1 2

$n = 6 \implies d = 6$





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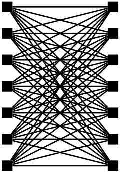
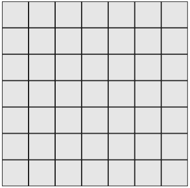
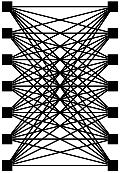
positions: 1 2

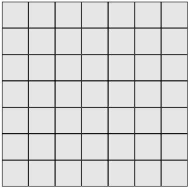
$n = 7 \implies d = 7$





Channel Model and Decoding





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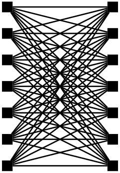
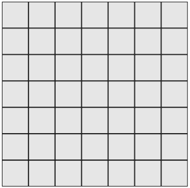
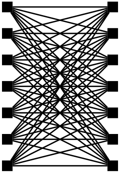




- Each CN corresponds to t-erasure correcting component code

Channel Model and Decoding



0	1	0	1	0	1	0
0	1	0	1	1	0	1
0	1	0	1	0	1	0
1	1	1	0	1	1	0
0	0	1	0	0	0	1
1	0	0	0	1	1	1
0	1	0	0	0	1	1

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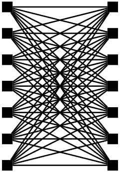

0	1	0	1	0	1	0
0	1	0	1	1	0	1
0	1	0	1	0	1	0
1	1	1	0	1	1	0
0	0	1	0	0	0	1
1	0	0	0	1	1	1
0	1	0	0	0	1	1

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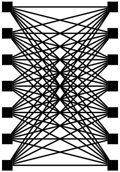
Channel Model and Decoding



0	?	0	?	0	1	?
?	1	0	1	1	0	1
0	1	0	?	0	?	?
1	1	1	0	1	1	0
0	0	1	0	0	0	1
1	0	?	?	1	1	?
0	1	0	?	0	1	1

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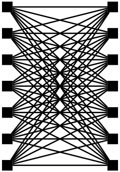
Channel Model and Decoding



0	?	0	?	0	1	?
?	1	0	1	1	0	1
0	1	0	?	0	?	?
1	1	1	0	1	1	0
0	0	1	0	0	0	1
1	0	?	?	1	1	?
0	1	0	?	0	1	1

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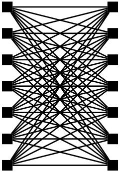

0	?	0	?	0	1	?
?	1	0	1	1	0	1
0	1	0	?	0	?	?
1	1	1	0	1	1	0
0	0	1	0	0	0	1
1	0	?	?	1	1	?
0	1	0	?	0	1	1

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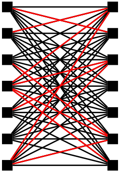

0	?	0	?	0	1	?
?	1	0	1	1	0	1
0	1	0	?	0	?	?
1	1	1	0	1	1	0
0	0	1	0	0	0	1
1	0	?	?	1	1	?
0	1	0	?	0	1	1

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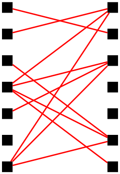

0	?	0	?	0	1	?
?	1	0	1	1	0	1
0	1	0	?	0	?	?
1	1	1	0	1	1	0
0	0	1	0	0	0	1
1	0	?	?	1	1	?
0	1	0	?	0	1	1

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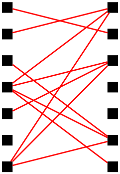

0	?	0	?	0	1	?
?	1	0	1	1	0	1
0	1	0	?	0	?	?
1	1	1	0	1	1	0
0	0	1	0	0	0	1
1	0	?	?	1	1	?
0	1	0	?	0	1	1

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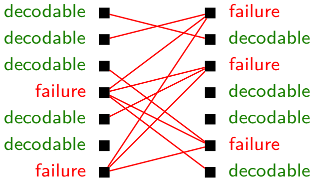
Channel Model and Decoding

1st iteration $(t = 2)$



0	?	0	?	0	1	?
?	1	0	1	1	0	1
0	1	0	?	0	?	?
1	1	1	0	1	1	0
0	0	1	0	0	0	1
1	0	?	?	1	1	?
0	1	0	?	0	1	1

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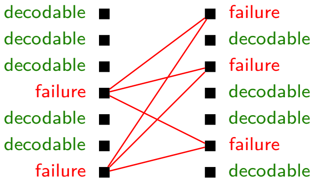

0	1	0	?	0	1	?
0	1	0	1	1	0	1
0	1	0	?	0	1	?
1	1	1	0	1	1	0
0	0	1	0	0	0	1
1	0	0	?	1	1	?
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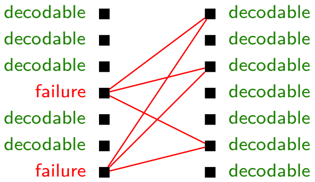
Channel Model and Decoding

2nd iteration $(t = 2)$



0	1	0	?	0	1	?
0	1	0	1	1	0	1
0	1	0	?	0	1	?
1	1	1	0	1	1	0
0	0	1	0	0	0	1
1	0	0	?	1	1	?
0	1	0	0	0	1	1

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Channel Model and Decoding

2nd iteration $(t = 2)$

decodable ■	■ decodable
decodable ■	■ decodable
decodable ■	■ decodable
failure ■	■ decodable
decodable ■	■ decodable
decodable ■	■ decodable
failure ■	■ decodable

0	1	0	1	0	1	0
0	1	0	1	1	0	1
0	1	0	1	0	1	0
1	1	1	0	1	1	0
0	0	1	0	0	0	1
1	0	0	0	1	1	1
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Density Evolution

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- What happens asymptotically for $n \to \infty$?

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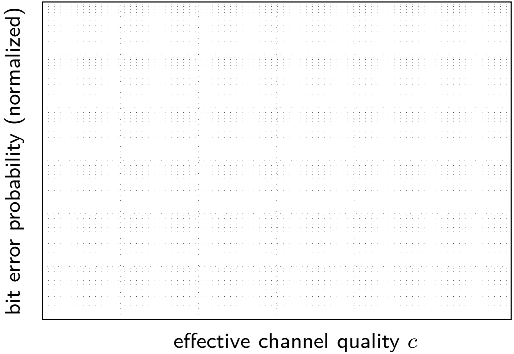

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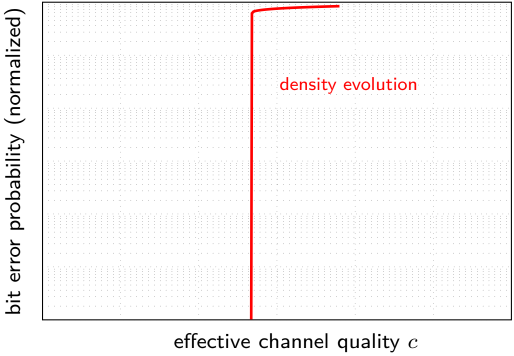

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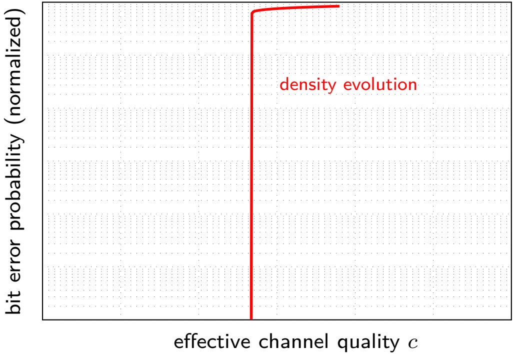

$$\boldsymbol{x}^{(\ell)} = \boldsymbol{\Psi}_{\geq t}(c\boldsymbol{B}\boldsymbol{x}^{(\ell-1)})$$

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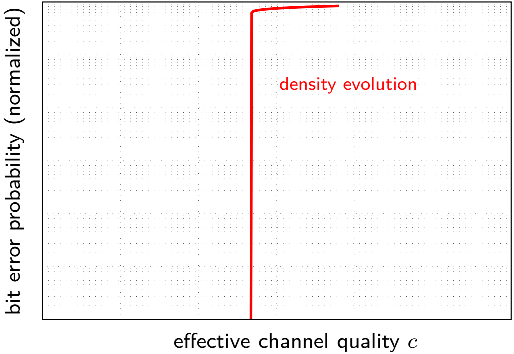

initial condition
$\boldsymbol{x}^{(0)} = (1, \dots, 1)$

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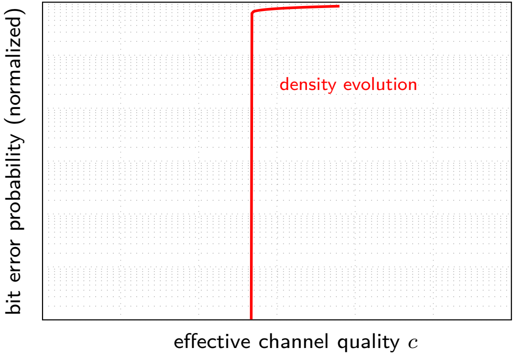

$B \triangleq \gamma \eta$

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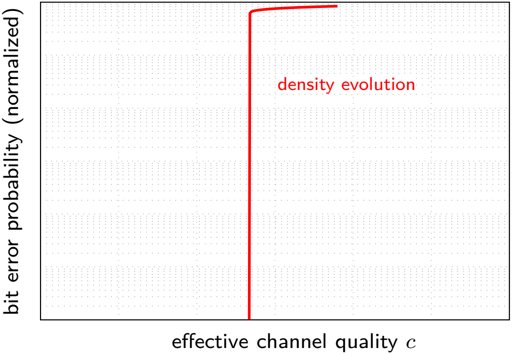

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element-wise application of $\Psi_{\geq t}(x) \triangleq 1 - \sum_{i=0}^{t-1} \frac{x^i}{i!} e^{-x}$

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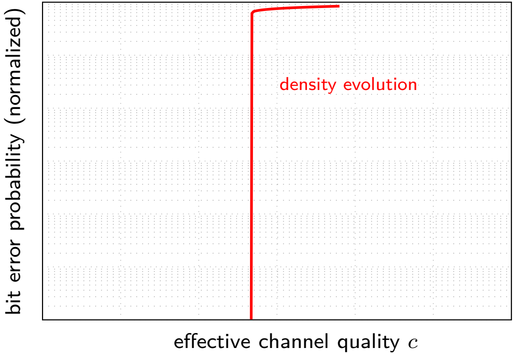

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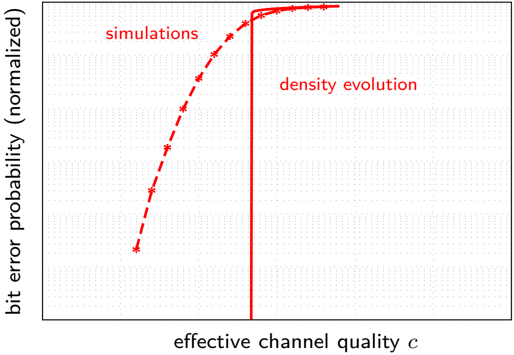

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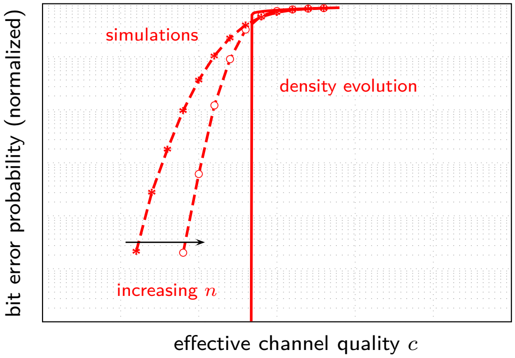

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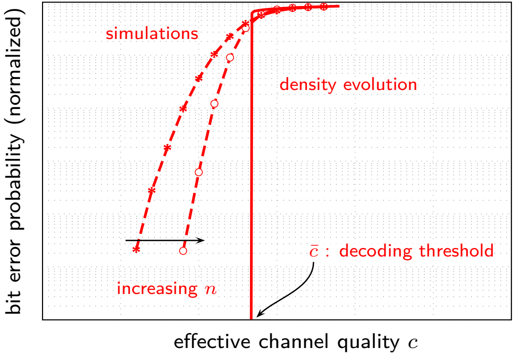

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Spatially-Coupled Product Codes

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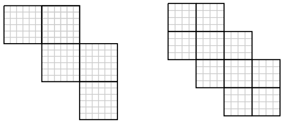
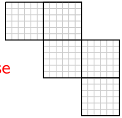
Deterministic

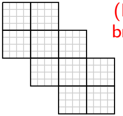


Spatially-Coupled Product Codes

Deterministic

staircase code



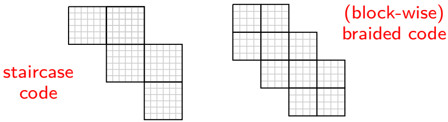


(block-wise) braided code

“convolutional-like” structure

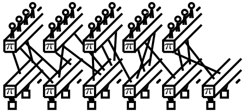
Spatially-Coupled Product Codes

Deterministic



"convolutional-like"
structure

Ensemble-Based [Jian et al., 2012]



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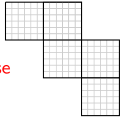
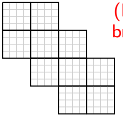
staircase code





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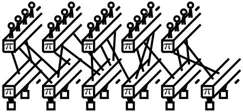
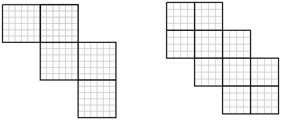


capacity-achieving at high rates over the binary symmetric channel

Spatially-Coupled Product Codes

Deterministic



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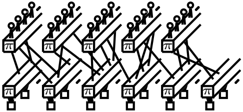

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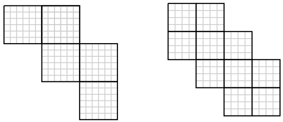

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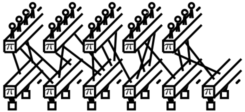
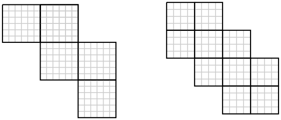


$$\boldsymbol{x}^{(\ell)} = \boldsymbol{\Psi}_{\geq t}(c\tilde{\boldsymbol{B}}\boldsymbol{x}^{(\ell-1)})$$

Spatially-Coupled Product Codes

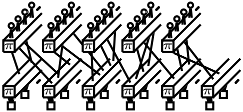
Deterministic



$$\boldsymbol{x}^{(\ell)} = \boldsymbol{\Psi}_{\geq t}(c\boldsymbol{B}\boldsymbol{x}^{(\ell-1)})$$

$$(\boldsymbol{B} = \gamma\boldsymbol{\eta})$$

Ensemble-Based [Jian et al., 2012]



$$\boldsymbol{x}^{(\ell)} = \boldsymbol{\Psi}_{\geq t}(c\tilde{\boldsymbol{B}}\boldsymbol{x}^{(\ell-1)})$$

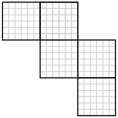
$$(\tilde{\boldsymbol{B}} = \boldsymbol{A}^{\intercal}\boldsymbol{A})$$

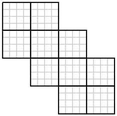
$$\boldsymbol{A} = \frac{1}{w}\begin{pmatrix} 1 & 1 & \cdots & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & \cdots & 1 & 0 & 0 \\ 0 & 0 & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & 0 & 1 & 1 & \cdots & 1 \end{pmatrix}$$

w (coupling width)

Spatially-Coupled Product Codes

Deterministic





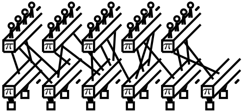
$$\boldsymbol{x}^{(\ell)} = \boldsymbol{\Psi}_{\geq t}(c\boldsymbol{B}\boldsymbol{x}^{(\ell-1)})$$

$(\boldsymbol{B} = \gamma\boldsymbol{\eta})$

$$\frac{1}{2}\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \quad \frac{1}{3}\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix},$$

staircase | braided (simplified)

Ensemble-Based [Jian et al., 2012]



$$\boldsymbol{x}^{(\ell)} = \boldsymbol{\Psi}_{\geq t}(c\tilde{\boldsymbol{B}}\boldsymbol{x}^{(\ell-1)})$$

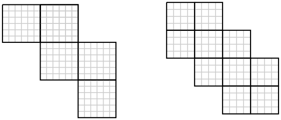
$(\tilde{\boldsymbol{B}} = \boldsymbol{A}^\intercal \boldsymbol{A})$

$$\frac{1}{4}\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}, \quad \frac{1}{9}\begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 2 & 2 & 1 & 0 & 0 \\ 1 & 2 & 3 & 2 & 1 & 0 \\ 0 & 1 & 2 & 3 & 2 & 1 \\ 0 & 0 & 1 & 2 & 2 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix}$$

$w = 2$ | $w = 3$

Spatially-Coupled Product Codes

Deterministic



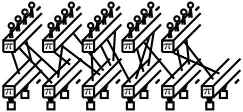
$$\boldsymbol{x}^{(\ell)} = \boldsymbol{\Psi}_{\geq t}(c\boldsymbol{B}\boldsymbol{x}^{(\ell-1)})$$

$(\boldsymbol{B} = \gamma\boldsymbol{\eta})$

$$\frac{1}{2}\begin{pmatrix} 0&1&0&0&0&0\\ 1&0&1&0&0&0\\ 0&1&0&1&0&0\\ 0&0&1&0&1&0\\ 0&0&0&1&0&1\\ 0&0&0&0&1&0 \end{pmatrix}, \quad \frac{1}{3}\begin{pmatrix} 1&1&0&0&0&0\\ 1&1&1&0&0&0\\ 0&1&1&1&0&0\\ 0&0&1&1&1&0\\ 0&0&0&1&1&1\\ 0&0&0&0&1&1 \end{pmatrix},$$

staircase — braided (simplified)

Ensemble-Based [Jian et al., 2012]



$$\boldsymbol{x}^{(\ell)} = \boldsymbol{\Psi}_{\geq t}(c\tilde{\boldsymbol{B}}\boldsymbol{x}^{(\ell-1)})$$

$(\tilde{\boldsymbol{B}} = \boldsymbol{A}^{\intercal}\boldsymbol{A})$

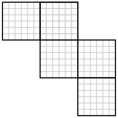
$$\frac{1}{4}\begin{pmatrix} 1&1&0&0&0&0\\ 1&2&1&0&0&0\\ 0&1&2&1&0&0\\ 0&0&1&2&1&0\\ 0&0&0&1&2&1\\ 0&0&0&0&1&1 \end{pmatrix}, \quad \frac{1}{9}\begin{pmatrix} 1&1&1&0&0&0\\ 1&2&2&1&0&0\\ 1&2&3&2&1&0\\ 0&1&2&3&2&1\\ 0&0&1&2&2&1\\ 0&0&0&1&1&1 \end{pmatrix}$$

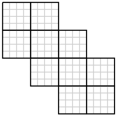
$w = 2$ — $w = 3$

- Equations have the same form, but different averaging matrices $\boldsymbol{B}$ and $\tilde{\boldsymbol{B}}$

Spatially-Coupled Product Codes

Deterministic





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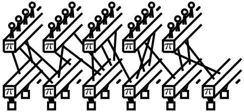
$$\frac{1}{2}\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \quad \frac{1}{3}\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix},$$

staircase | braided (simplified)

Ensemble-Based [Jian et al., 2012]



$$\boldsymbol{x}^{(\ell)} = \boldsymbol{\Psi}_{\geq t}(c\tilde{\boldsymbol{B}}\boldsymbol{x}^{(\ell-1)})$$

$$(\tilde{\boldsymbol{B}} = \boldsymbol{A}^\intercal\boldsymbol{A})$$

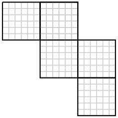
$$\frac{1}{4}\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}, \quad \frac{1}{9}\begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 2 & 2 & 1 & 0 & 0 \\ 1 & 2 & 3 & 2 & 1 & 0 \\ 0 & 1 & 2 & 3 & 2 & 1 \\ 0 & 0 & 1 & 2 & 2 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix}$$

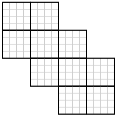
$w = 2$ | $w = 3$

- Equations have the same form, but different averaging matrices $\boldsymbol{B}$ and $\tilde{\boldsymbol{B}}$
- One can show that ensemble performance can be “emulated”

Spatially-Coupled Product Codes

Deterministic





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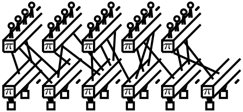
$(\boldsymbol{B} = \gamma\boldsymbol{\eta})$

$$\frac{1}{2}\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \quad \frac{1}{3}\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix},$$

staircase — braided (simplified)

Ensemble-Based [Jian et al., 2012]



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$w = 2$ — $w = 3$

- Equations have the same form, but different averaging matrices $\boldsymbol{B}$ and $\tilde{\boldsymbol{B}}$
- One can show that ensemble performance can be "emulated"
- $\implies$ ensemble threshold bounds in [Jian et al., 2012] apply to deterministic codes!

Symmetric Generalized Product Codes

Symmetric Generalized Product Codes

- So far, $\eta_{i,i} = 0$ for all $i \in \{1, 2, \ldots, L\}$. What about $\eta_{i,i} = 1$?

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

product code

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

staircase code

$$\begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \end{pmatrix}$$

(block-wise) braided code

Symmetric Generalized Product Codes

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Example: $L = 1$, $\boldsymbol{\eta} = 1$, and $\gamma = 1$

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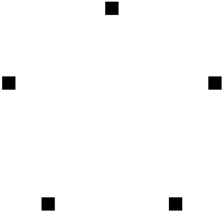
Example: $L = 1$, $\boldsymbol{\eta} = 1$, and $\gamma = 1$

$n = 5 \implies d = 5$



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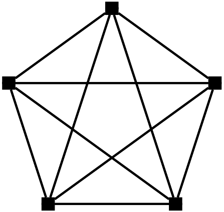

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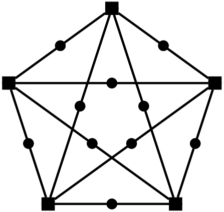

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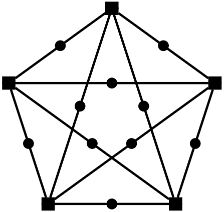

array representation?

Symmetric Generalized Product Codes

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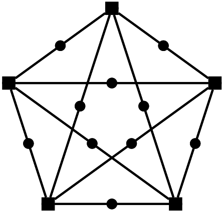

0	$*$	$*$	$*$	$*$
c_1	0	$*$	$*$	$*$
c_2	c_3	0	$*$	$*$
c_4	c_5	c_6	0	$*$
c_7	c_8	c_9	c_{10}	0

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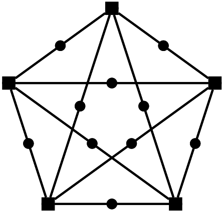

0	*	*	*	*
c_1	0	*	*	*
c_2	c_3	0	*	*
c_4	c_5	c_6	0	*
c_7	c_8	c_9	c_{10}	0

$*$ = not used

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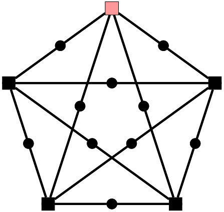

0	*	*	*	*
c_1	0	*	*	*
c_2	c_3	0	*	*
c_4	c_5	c_6	0	*
c_7	c_8	c_9	c_{10}	0

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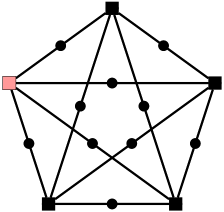

0	$*$	$*$	$*$	$*$
c_1	0	$*$	$*$	$*$
c_2	c_3	0	$*$	$*$
c_4	c_5	c_6	0	$*$
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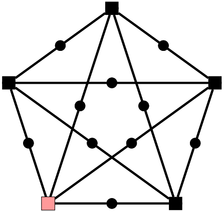

0	$*$	$*$	$*$	$*$
c_1	0	$*$	$*$	$*$
c_2	c_3	0	$*$	$*$
c_4	c_5	c_6	0	$*$
c_7	c_8	c_9	c_{10}	0

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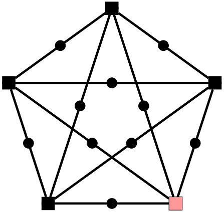

0	$*$	$*$	$*$	$*$
c_1	0	$*$	$*$	$*$
c_2	c_3	0	$*$	$*$
c_4	c_5	c_6	0	$*$
c_7	c_8	c_9	c_{10}	0

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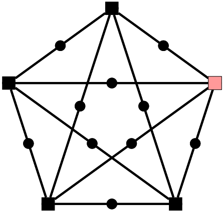

0	$*$	$*$	$*$	$*$
c_1	0	$*$	$*$	$*$
c_2	c_3	0	$*$	$*$
c_4	c_5	c_6	0	$*$
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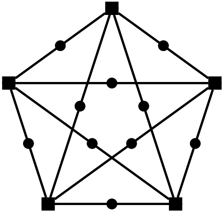

0	*	*	*	*
c_1	0	*	*	*
c_2	c_3	0	*	*
c_4	c_5	c_6	0	*
c_7	c_8	c_9	c_{10}	0

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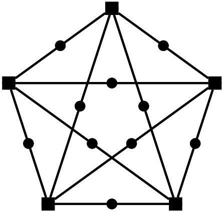

0	c_1	c_2	c_4	c_7
c_1	0	c_3	c_5	c_8
c_2	c_3	0	c_6	c_9
c_4	c_5	c_6	0	c_{10}
c_7	c_8	c_9	c_{10}	0

symmetric array

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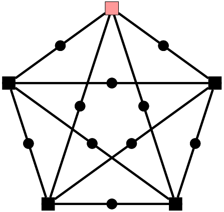

0	c_1	c_2	c_4	c_7
c_1	0	c_3	c_5	c_8
c_2	c_3	0	c_6	c_9
c_4	c_5	c_6	0	c_{10}
c_7	c_8	c_9	c_{10}	0

symmetric array

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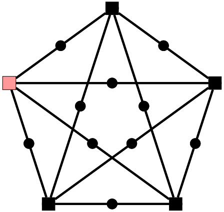


0	c_1	c_2	c_4	c_7
c_1	0	c_3	c_5	c_8
c_2	c_3	0	c_6	c_9
c_4	c_5	c_6	0	c_{10}
c_7	c_8	c_9	c_{10}	0

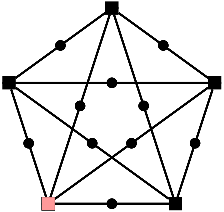
symmetric array

Symmetric Generalized Product Codes

- So far, $\eta_{i,i} = 0$ for all $i \in \{1, 2, \ldots, L\}$. What about $\eta_{i,i} = 1$?

Example: $L = 1$, $\boldsymbol{\eta} = 1$, and $\gamma = 1$

$n = 5 \implies d = 5$



0	c_1	c_2	c_4	c_7
c_1	0	c_3	c_5	c_8
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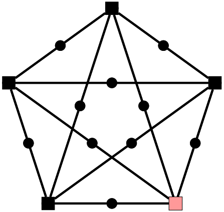

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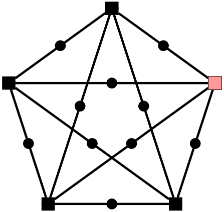

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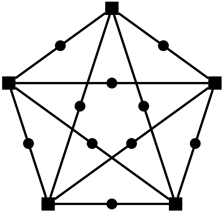

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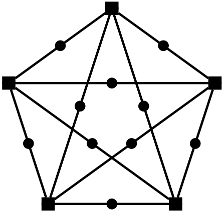

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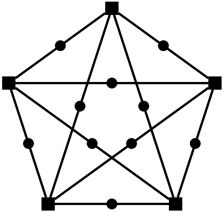

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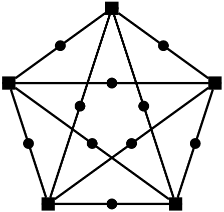

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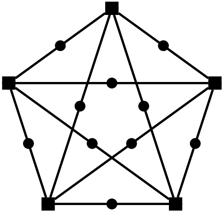

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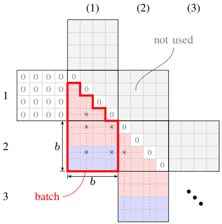
- A half-product code has the same threshold as a product code, but less than half the block length
- Half-braided codes can outperform staircase and braided codes in the waterfall region, at a lower error floor and decoding delay [Häger et al., 2016]



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Thank you!

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