Density Evolution for Deterministic Generalized Product Codes with Higher-Order Modulation

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In This Talk . . .

- Deterministic code construction that recovers product codes, staircase codes, and block-wise braided codes as special cases
- Rigorous density evolution analysis over parallel BECs
- Application: Bit mapper (interleaver) optimization for coded modulation

Introduction: Product Codes and Staircase Codes

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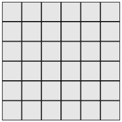
rectangular array [Elias, 1954]



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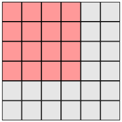

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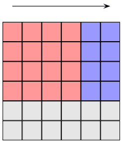

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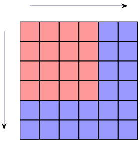

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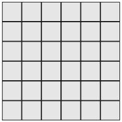

each row/column is a codeword in some component code

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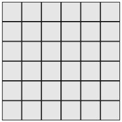

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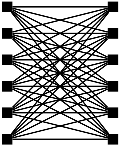

constraint node (CN) degree = component code length

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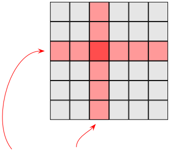

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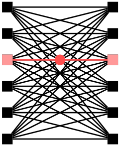

edge = degree-2 variable node (VN)

constraint node (CN) degree = component code length

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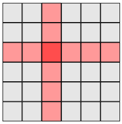

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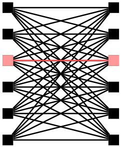

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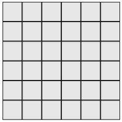

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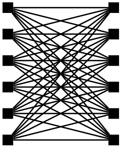

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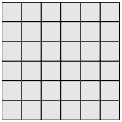

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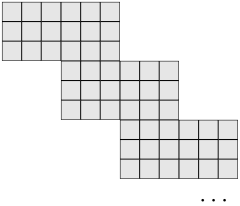

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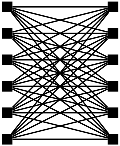

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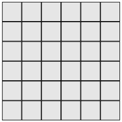

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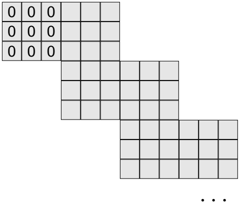

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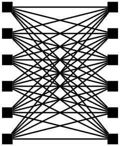

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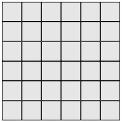

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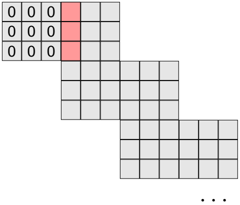

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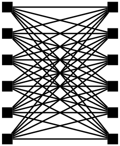

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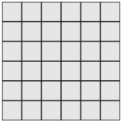

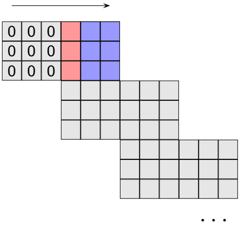

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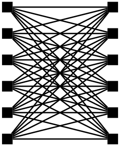

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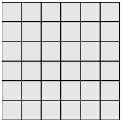

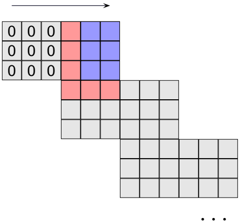

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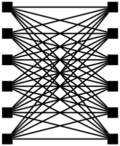

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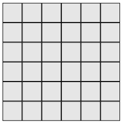

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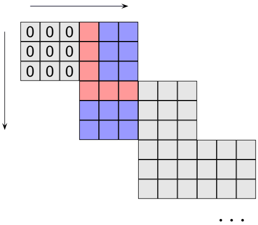

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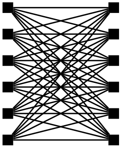

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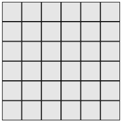

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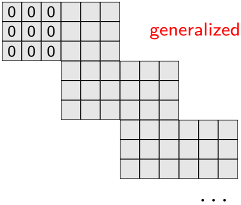
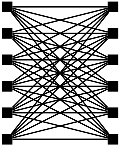


generalized product code (GPC)

Tanner graph



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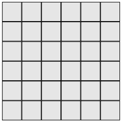

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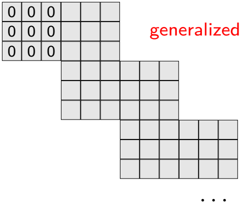

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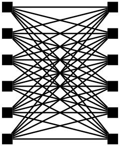

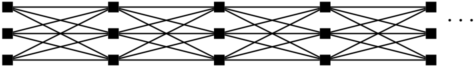

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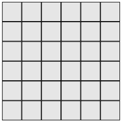

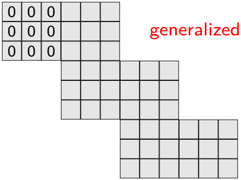

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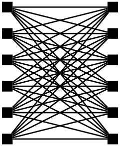
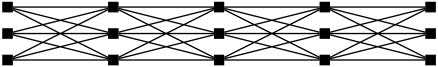


spatially-coupled code



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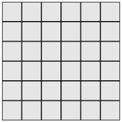

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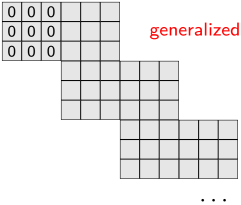

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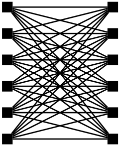

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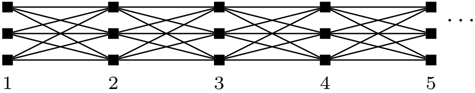
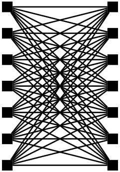
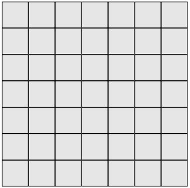


positions:

- Deterministic codes with fixed and structured Tanner graph

Iterative Bounded-Distance Decoding





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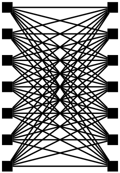

0	1	0	1	0	1	0
0	1	0	1	1	0	1
0	1	0	1	0	1	0
1	1	1	0	1	1	0
0	0	1	0	0	0	1
1	0	0	0	1	1	1
0	1	0	0	0	1	1

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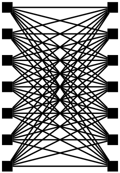
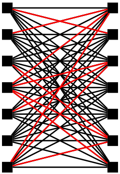


0	?	0	?	0	1	?
?	1	0	1	1	0	1
0	1	0	?	0	?	?
1	1	1	0	1	1	0
0	0	1	0	0	0	1
1	0	?	?	1	1	?
0	1	0	?	0	1	1

- Codeword transmission over the BEC with erasure probability p

Iterative Bounded-Distance Decoding



0	?	0	?	0	1	?
?	1	0	1	1	0	1
0	1	0	?	0	?	?
1	1	1	0	1	1	0
0	0	1	0	0	0	1
1	0	?	?	1	1	?
0	1	0	?	0	1	1

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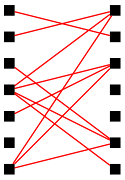

0	?	0	?	0	1	?
?	1	0	1	1	0	1
0	1	0	?	0	?	?
1	1	1	0	1	1	0
0	0	1	0	0	0	1
1	0	?	?	1	1	?
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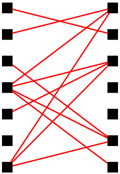
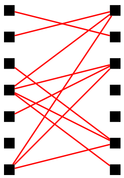

residual graph

0	?	0	?	0	1	?
?	1	0	1	1	0	1
0	1	0	?	0	?	?
1	1	1	0	1	1	0
0	0	1	0	0	0	1
1	0	?	?	1	1	?
0	1	0	?	0	1	1

- Codeword transmission over the BEC with erasure probability p
- Each CN corresponds to t-erasure correcting component code

Iterative Bounded-Distance Decoding



residual graph

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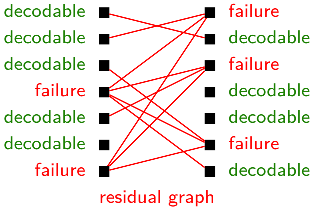
?	1	0	1	1	0	1
0	1	0	?	0	?	?
1	1	1	0	1	1	0
0	0	1	0	0	0	1
1	0	?	?	1	1	?
0	1	0	?	0	1	1

- Codeword transmission over the BEC with erasure probability p
- Each CN corresponds to t-erasure correcting component code
- ℓ iterations of bounded-distance decoding $=$ peeling of vertices with degree $\leq t$ (in parallel)

Iterative Bounded-Distance Decoding

1st iteration $(t = 2)$



0	?	0	?	0	1	?
?	1	0	1	1	0	1
0	1	0	?	0	?	?
1	1	1	0	1	1	0
0	0	1	0	0	0	1
1	0	?	?	1	1	?
0	1	0	?	0	1	1

- Codeword transmission over the BEC with erasure probability p
- Each CN corresponds to t-erasure correcting component code
- ℓ iterations of bounded-distance decoding $=$ peeling of vertices with degree $\leq t$ (in parallel)

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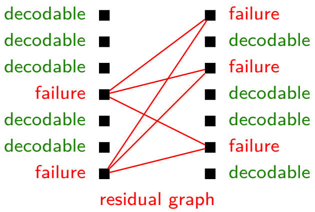

0	1	0	?	0	1	?
0	1	0	1	1	0	1
0	1	0	?	0	1	?
1	1	1	0	1	1	0
0	0	1	0	0	0	1
1	0	0	?	1	1	?
0	1	0	0	0	1	1

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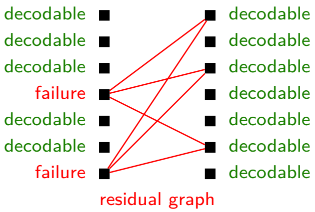
Iterative Bounded-Distance Decoding

2nd iteration $(t = 2)$



0	1	0	?	0	1	?
0	1	0	1	1	0	1
0	1	0	?	0	1	?
1	1	1	0	1	1	0
0	0	1	0	0	0	1
1	0	0	?	1	1	?
0	1	0	0	0	1	1

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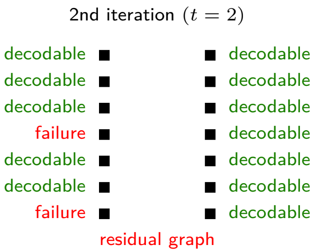
Iterative Bounded-Distance Decoding



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0	1	0	1	0	1	0
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0	0	1	0	0	0	1
1	0	0	0	1	1	1
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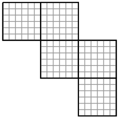
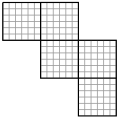
Asymptotic Performance Prediction

Asymptotic Performance Prediction



- Example: staircase code with a fixed component code

Asymptotic Performance Prediction



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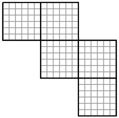
Asymptotic Performance Prediction



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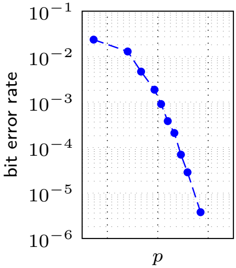
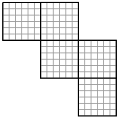


Asymptotic Performance Prediction



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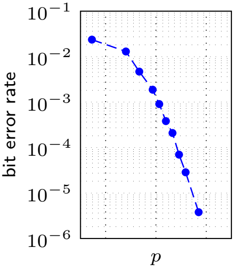

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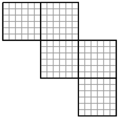

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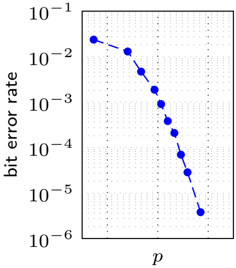

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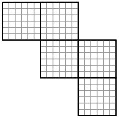

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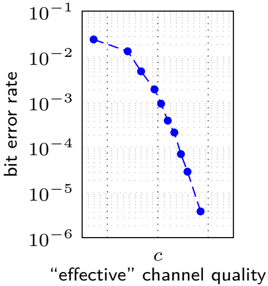

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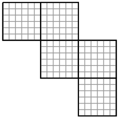

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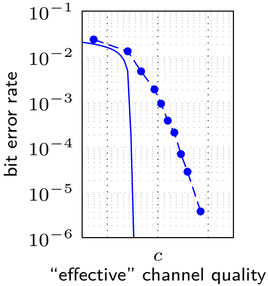

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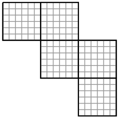

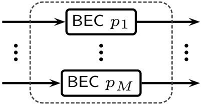

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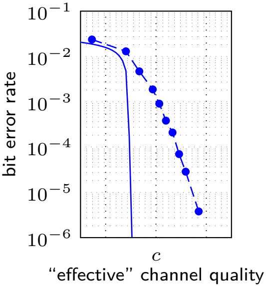
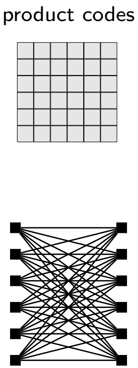
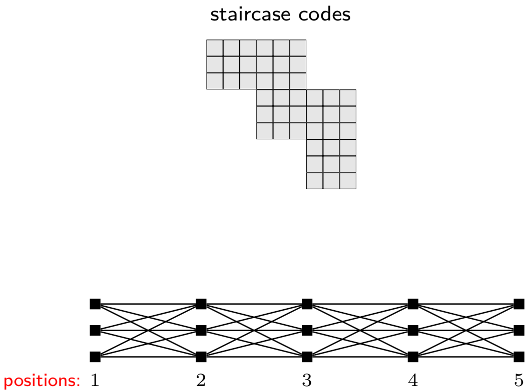


Main contribution

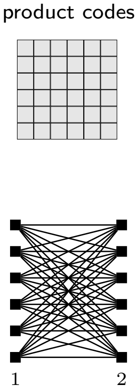
Generalization to parallel BECs with different erasure probabilities

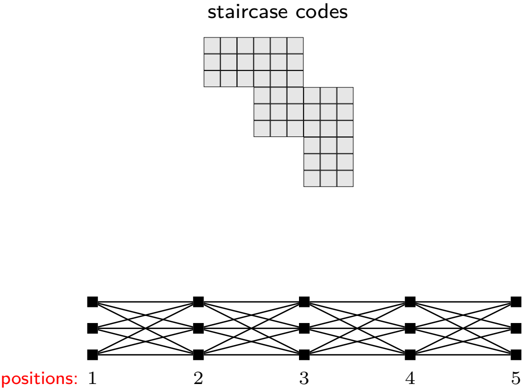
Parametrized Construction of Generalized Product Codes





Parametrized Construction of Generalized Product Codes





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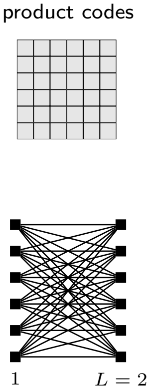

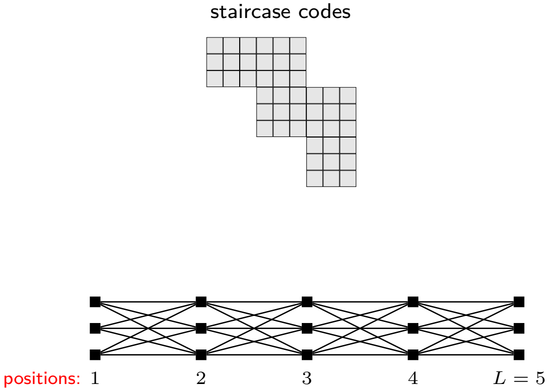

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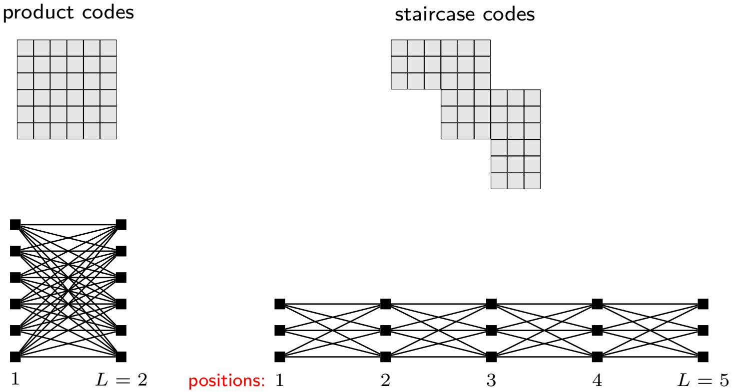
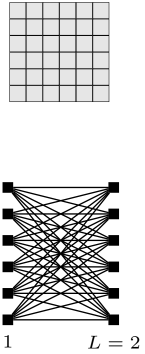


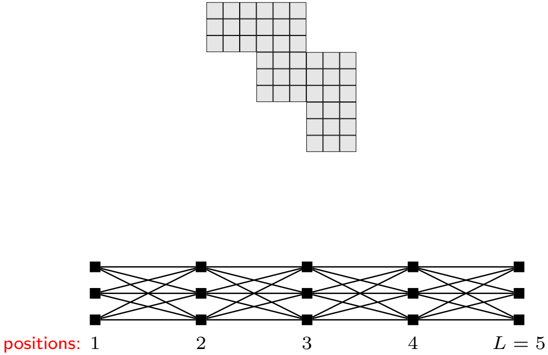
$\boldsymbol{\eta}$: symmetric $L \times L$ matrix that defines graph connectivity

Parametrized Construction of Generalized Product Codes

product codes



staircase codes

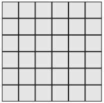


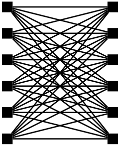
$$\boldsymbol{\eta} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$\boldsymbol{\eta}$: symmetric $L \times L$ matrix that defines graph connectivity

Parametrized Construction of Generalized Product Codes

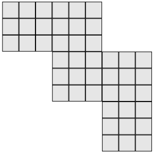
product codes

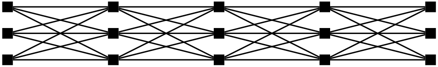




$1 \qquad L = 2$

staircase codes





positions: $1 \qquad 2 \qquad 3 \qquad 4 \qquad L = 5$

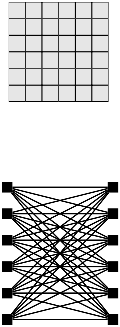
$$\boldsymbol{\eta} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\boldsymbol{\eta} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

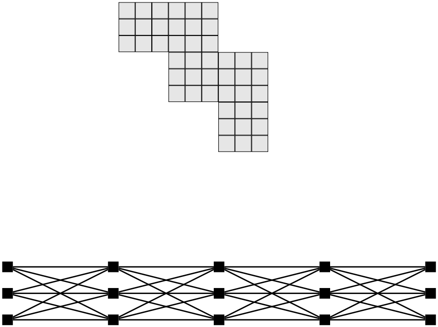
$\boldsymbol{\eta}$: symmetric $L \times L$ matrix that defines graph connectivity

Parametrized Construction of Generalized Product Codes

product codes

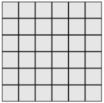


staircase codes



Parametrized Construction of Generalized Product Codes

product codes



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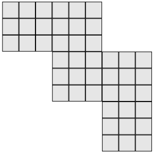
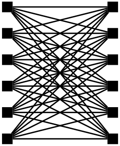
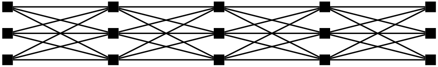


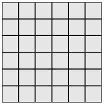


n: "problem size", proportional to the number of constraint nodes

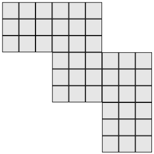


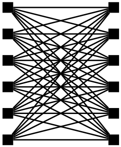
Parametrized Construction of Generalized Product Codes

product codes



staircase codes





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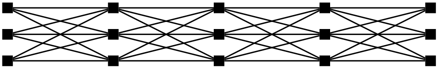
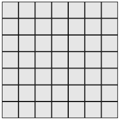


increasing n

Parametrized Construction of Generalized Product Codes

product codes



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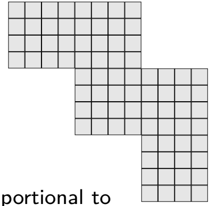

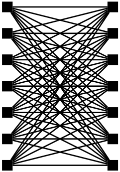

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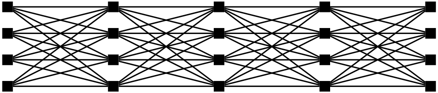
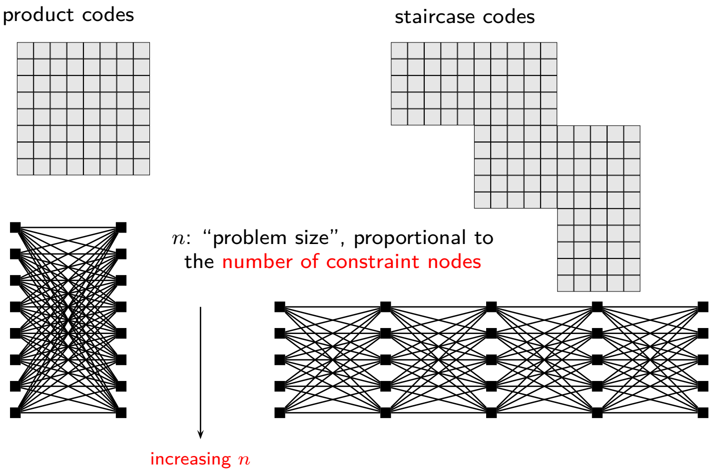


increasing n

Parametrized Construction of Generalized Product Codes



Component Code Mixtures

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- For each position $i \in \{1, 2, \ldots, L\}$, let $\boldsymbol{\tau}(i) = (\tau_1(i), \tau_2(i), \ldots, \tau_{t_{\mathsf{max}}}(i))^\intercal$ be a probability vector/distribution ($\sum_t \tau_t(i) = 1$ and $\tau_t(i) \geq 0$ for all i)

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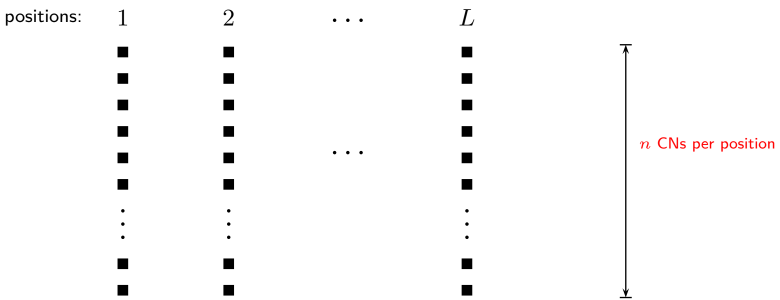

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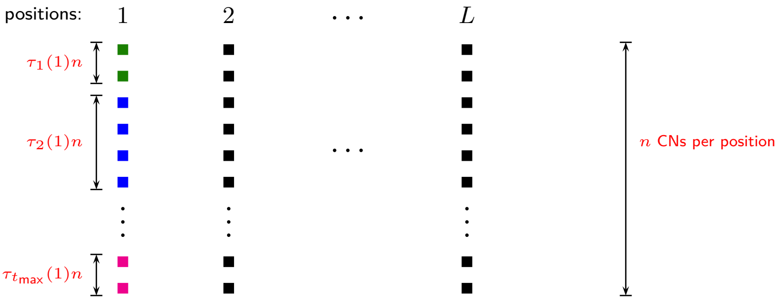

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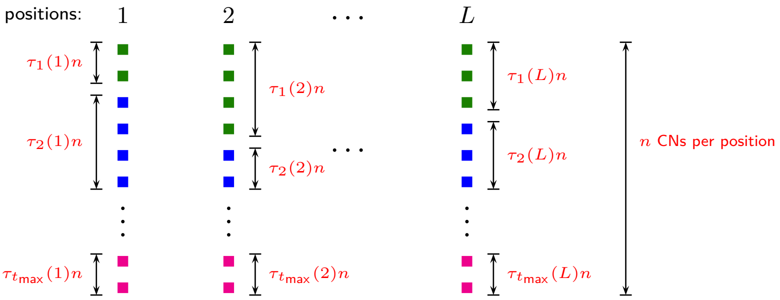

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  - $\tau_{t_1}(1) = 0.2$, $\tau_{t_2}(1) = 0.3$, $\tau_{t_3}(1) = 0.5$
  - $\tau_{t_4}(2) = 0.3$, $\tau_{t_5}(2) = 0.7$

Component Code Mixtures

- For each position $i \in \{1, 2, \ldots, L\}$, let $\boldsymbol{\tau}(i) = (\tau_1(i), \tau_2(i), \ldots, \tau_{t_{\mathsf{max}}}(i))^\mathsf{T}$ be a probability vector/distribution ($\sum_t \tau_t(i) = 1$ and $\tau_t(i) \geq 0$ for all i)
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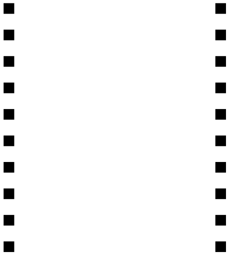

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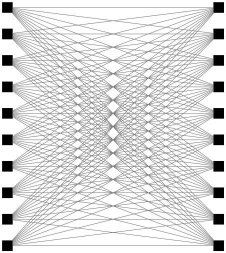

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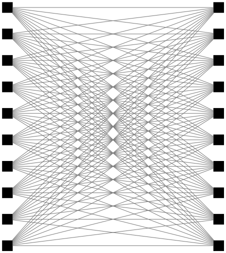

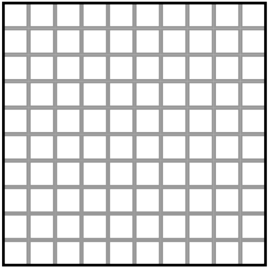

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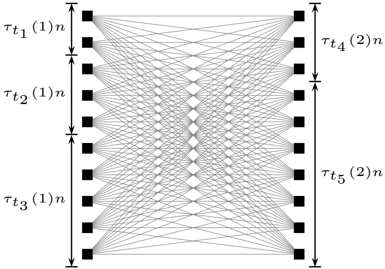

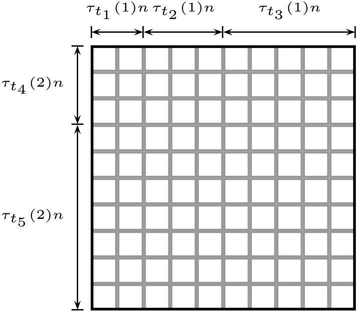

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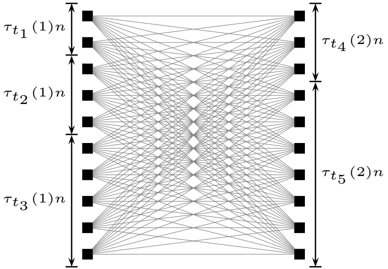
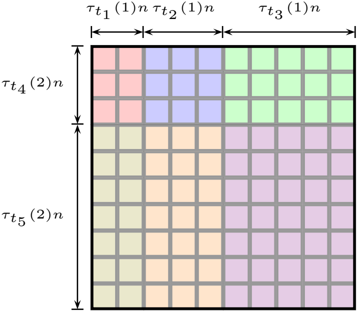
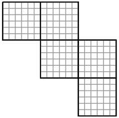
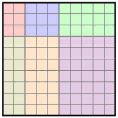


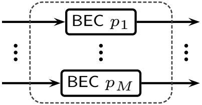


$K = 6$ distinct VN classes

Bit Mapper







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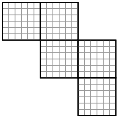

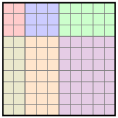

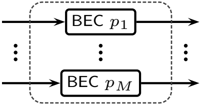

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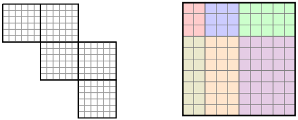

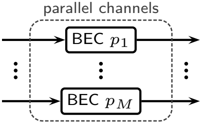

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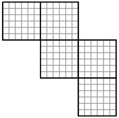

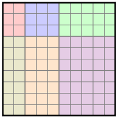

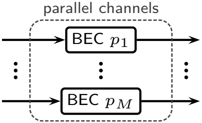

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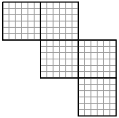

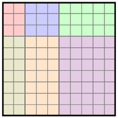

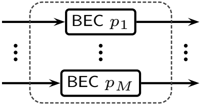

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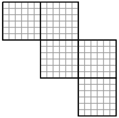

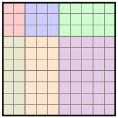

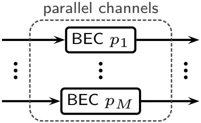

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- Example: baseline/uniform bit mapper, where $a_{k,q} = 1/M$ for all k, q
  $\implies$ all virtual BECs are the same

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Poisson tail probability $\Psi_{\geq t}(x) \triangleq 1 - \sum_{i=0}^{t-1}\frac{x^i}{i!}e^{-x}$

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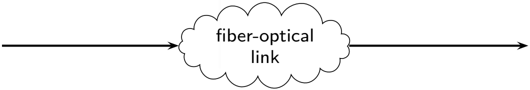
$$\bar{c} = \sup\{c > 0 \,|\, \lim_{\ell\to\infty} x_{i,t}^{(\ell)} = 0 \text{ for all } i, t\}$$

Spectrally-Efficient Communication



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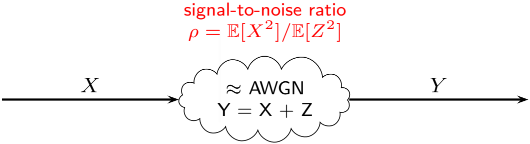

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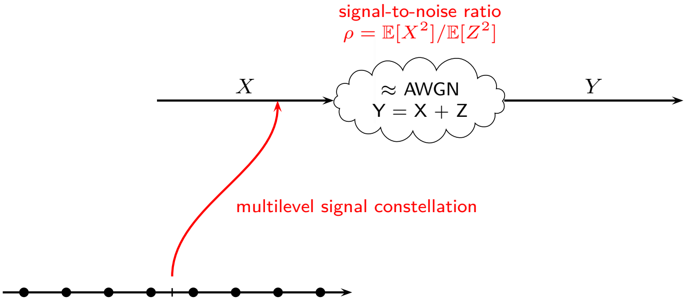

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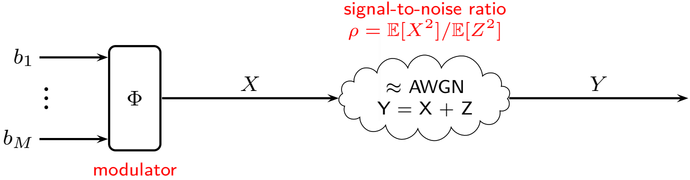

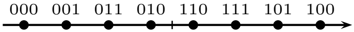

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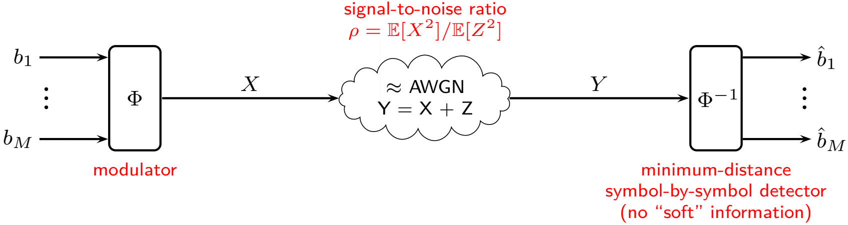

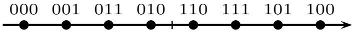

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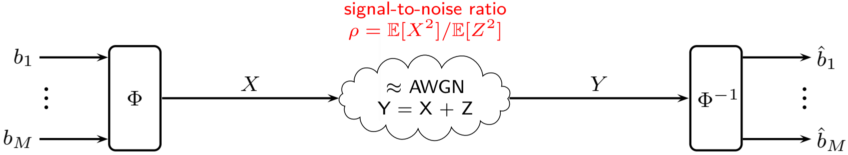

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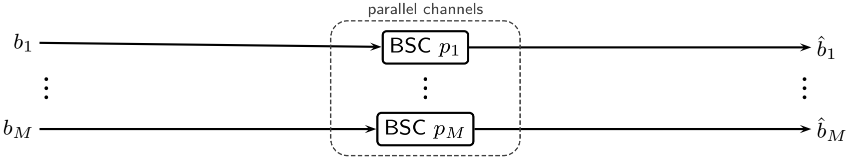

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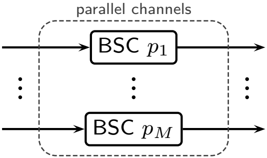

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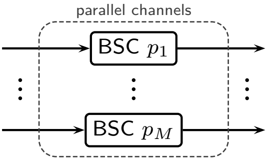

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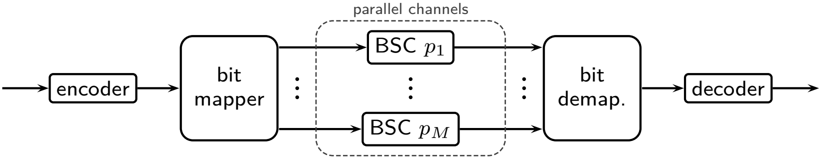
Spectrally-Efficient Communication



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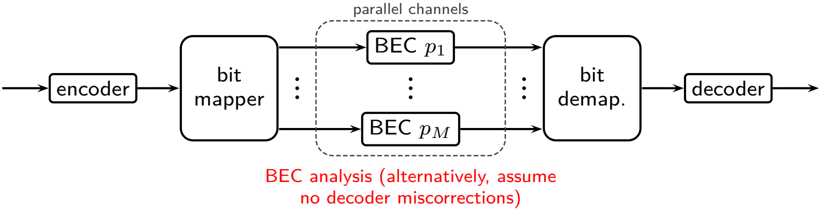

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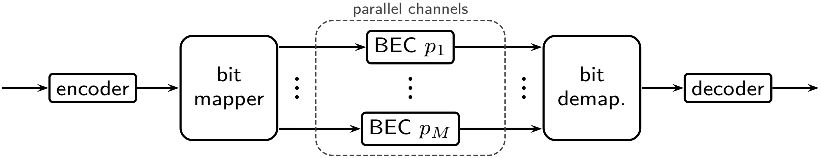

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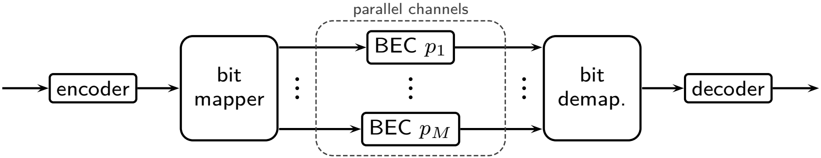

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- Example: threshold $\bar{c} = 10.63$ and $M = 4$. For $n = 1600$, waterfall region expected at $\rho = \bar{p}^{-1}(c/n) \approx 26.11\,\text{dB}$

Bit Mapper Optimization

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Problem Formulation ([Richter et al., 2007], [Cheng et al., 2012], . . .)

Optimize the bit mapper $\boldsymbol{A}$ for a given code and signal constellation

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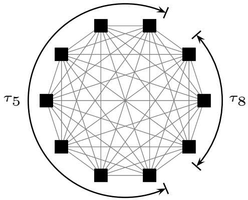

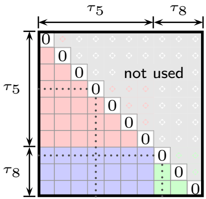

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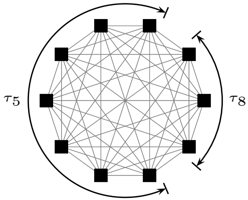

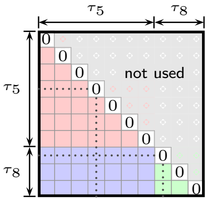

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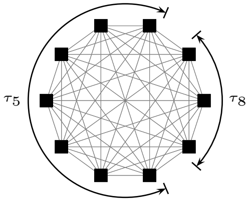

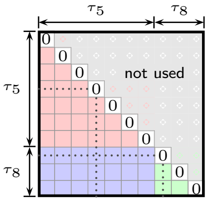

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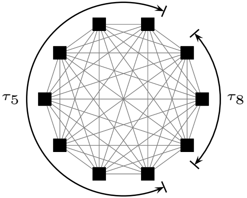

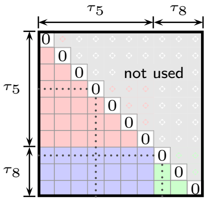

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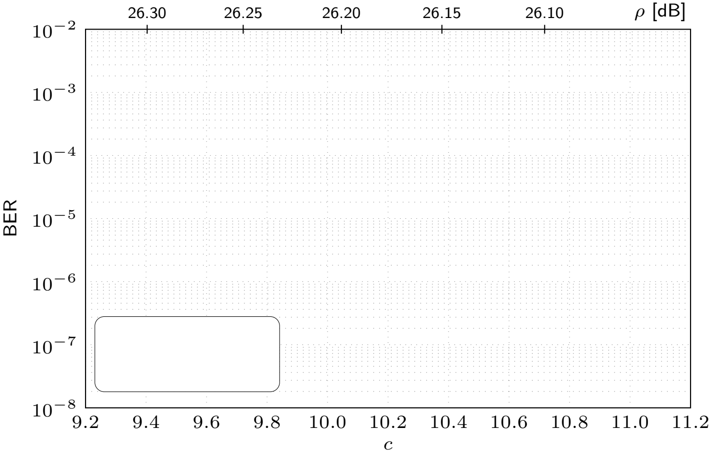
Results


26.30
26.25
26.20
26.15
26.10
ρ [dB]
BER
10^{-2}
10^{-3}
10^{-4}
10^{-5}
10^{-6}
10^{-7}
10^{-8}
9.2
9.4
9.6
9.8
10.0
10.2
10.4
10.6
10.8
11.0
11.2
c

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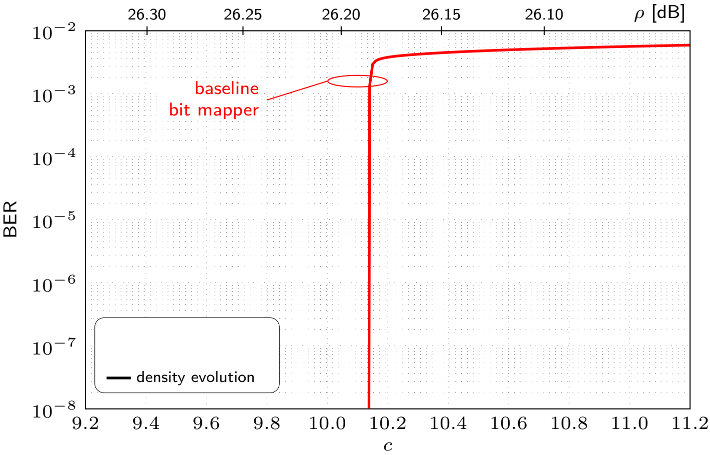

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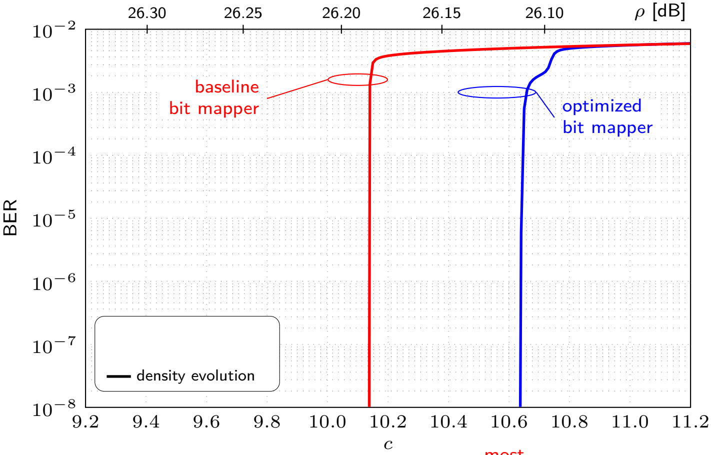
Results



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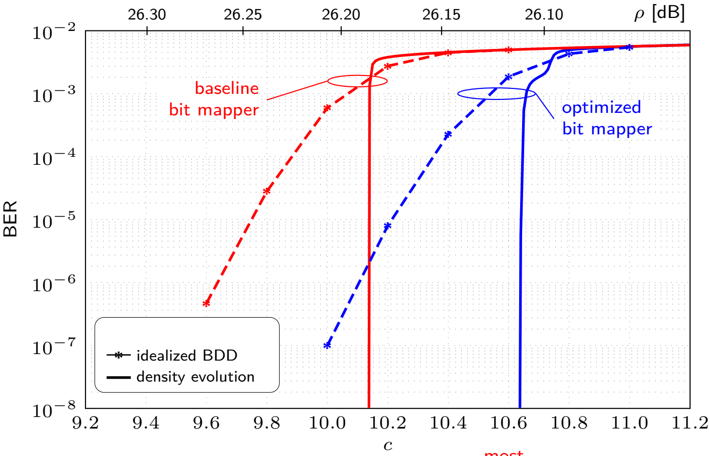

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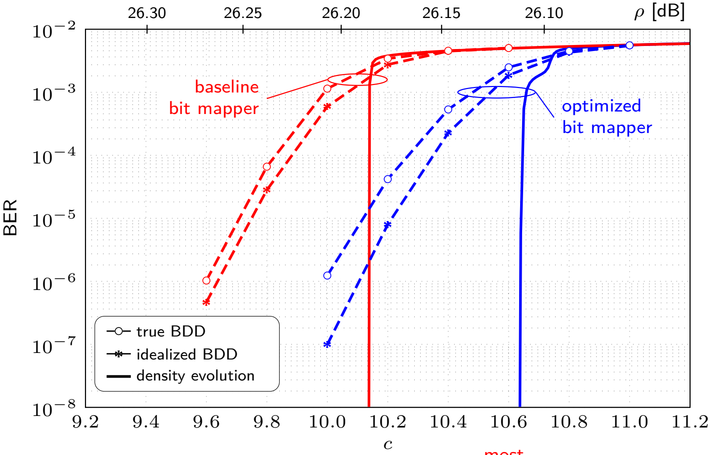

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Thank you!

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